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On the problem of optimal instant observations of the linear birth and death process

Alexander A. Butov *

Faculty of Mathematics and Information technologies of Ulyanovsk State University, Ulyanovsk, Russian Federation

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ABSTRACT

The problem of the optimal intensity of instant observations of the processes of birth and death with linear growth and immigration is considered. Results are obtained for the objective function calculated as an expected normalized linear function of the number of observations and an expected normalized quadratic form of the errors of estimation, given instant observations.

1. Introduction and main results

The processes of birth and death with linear growth and immigration have been extensively studied in the literature for many years. For example, see Karlin and McGregor (1958), Cavender (1978), Liu and Zhang (2011) and the references therein. Along with these approaches, the martingale methods were developed. At times, the martingale (or trajectory) methods can lead to interesting or nontrivial results for well-known processes. The correct approaches have been developed for random walks and for the processes of birth and death (see, e.g. Butov, 1991, 1994a,b). The aim of this note is to consider the optimal control problem for the intensity of Poisson-type observations of the processes of birth and death with linear growth and immigration for the case of a counting Poisson process in semimartingale terms. The proof of the result utilizes the semimartingale technique.

The linear function of the intensity as a cost of observations and the expected value of the quadratic form of errors of estimation as a cost of an error are calculated using a loss function.

Let $(\Omega, \mathcal{F}, \boldsymbol{P})$ be a probability space populated with a nondecreasing right-continuous family of σ-algebras $F = (\mathcal{F}_t)_{t\geq 0}$, complete with respect to $\boldsymbol{P}$ (i.e., the conditions of Dellacherie, 1972). We consider the following model. On the basis $\boldsymbol{B} = (\Omega, \mathcal{F}, F, \boldsymbol{P})$ a process $X = (X_t)_{t\geq 0}$ is a birth and death process with trajectories in the Skorokhod space, $X_t \in \boldsymbol{N}_0 = \{0, 1, 2, \dots\}$ and $\Delta X_t = X_t - X_{t-} \in \{-1, 0, 1\}$ (see, e.g. Butov, 1994b). The process X can be represented as a difference of two counting processes:

$$X = X_0 + B - D, \tag{1.1}$$

* Tel.: +7 9025820739.
E-mail address: butov.a.a@gmail.com.

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where $B = (B_t)_{t\geq 0}$ and $D = (D_t)_{t\geq 0}$ are the counting processes of the number of positive and negative jumps, respectively:

$$B_t = \sum_{0<s\leq t} I\{\Delta X_s = 1\}, \qquad D_t = \sum_{0<s\leq t} I\{\Delta X_s = -1\},$$

$$\sum_{0<s\leq t} \Delta B_s \cdot \Delta D_s = 0 \quad \text{for all } t \geq 0,$$

with the initial values $B_0 = D_0 = 0$ and $X_0 > 0$ (i.e., $X \in \boldsymbol{N} = \{1, 2, \ldots\}$). We suppose that submartingales B and D on $\boldsymbol{B}$ admit the well-known Doob–Meyer decomposition (see, e.g. Liptser and Shiryaev, 1989):

$$B_t = \tilde{B}_t + m_t^B,$$
$$D_t = \tilde{D}_t + m_t^D, \tag{1.2}$$

with the compensators

$$\tilde{B}_t = \int_0^t (\alpha + \beta \cdot X_s) ds,$$
$$\tilde{D}_t = \int_0^t \delta \cdot X_s\, ds, \tag{1.3}$$

and the square-integrable martingales $m^B = (m_t^B)_{t\geq 0}$ and $m^D = (m_t^D)_{t\geq 0}$ with the quadratic characteristics

$$\langle m^B \rangle_t = \tilde{B}_t, \qquad \langle m^D \rangle_t = \tilde{D}_t \quad \text{for all } t \geq 0. \tag{1.4}$$

We suppose that

$$\alpha > 0, \qquad \beta > 0, \qquad \delta > 0 \quad \text{and} \quad \delta > \beta. \tag{1.5}$$

Remark 1. The process X with the representation (1.1)–(1.4) under the assumptions of (1.5) is a well-known birth and death process with linear growth and immigration (and the compensator of the counting process of immigration is $\alpha \cdot t$).

For such a process, the conditions (1.5) are necessary and sufficient for the existence of a stationary and ergodic distribution of X and for a proper convergence of the distributions and expected values and variances of X to that of a stationary one ($X_t \xrightarrow{d} \xi$ as $t \to \infty$ for any initial distribution of X_0) (see, e.g. Liu and Zhang, 2011). We suppose, without loss of generality, that the distributions of X_0 and ξ coincide and that $X$is stationary. Consider on $\boldsymbol{B}$ a Poisson process $\pi = (\pi_t)_{t\geq 0}$ with a parameter $\lambda > 0$. Suppose that π and X are independent and that π is the counting process of the number of observations of X. The intensity of observation is equal to the parameter λ and the compensator of π *is* $\tilde{\pi}_t = \lambda \cdot t$. Then, the process of observations $Y = (Y_t)_{t\geq 0}$ can be written as

$$Y_t = Y_0 + \int_0^t (X_s - Y_{s-}) d\pi_s. \tag{1.6}$$

Because the processes π and X are independent, it then holds that

$$\sum_{0<s\leq t} \Delta \pi_s \cdot \Delta B_s = \sum_{0<s\leq t} \Delta \pi_s \cdot \Delta D_s = 0 \quad \text{for all } t \geq 0.$$

Let $\tau(k) = \inf\{t : t > 0, \pi_t \geq k\}$ for all $k \in \boldsymbol{N}$. Without loss of generality, we assume that the initial value of X is observable: $Y_0(\omega) = X_0(\omega)$ for all $\omega \in \Omega$; therefore, it is reasonable to let $\tau(0) = 0$. Then, for stopping times $\tau(k)$ and random variables $\theta(k) = \tau(k) - \tau(k-1)$ for all $k \in \boldsymbol{N}$, assume that $\boldsymbol{E}\theta(k) = 1/\lambda$, $\boldsymbol{D}\{\theta(k)\} = 1/\lambda^2$, $\boldsymbol{E}\tau(k) = k/\lambda$, and $\boldsymbol{D}\{\tau(k)\} = k/\lambda^2$, that the random variables $\theta(k)$ are independent and that their distribution density function $\rho(x)$ is equal to $\lambda \cdot \exp\{-\lambda \cdot x\}$ for $x \geq 0$ and 0 for $x < 0$.

Remark 2. From (1.6), it follows that values Y_t coincide with X_t at times $t = \tau(k)$ for all $k \in \boldsymbol{N}$. Hence $F_t^Y = \sigma\{X_{\tau(0)}, X_{\tau(1)} \cdot I\{\tau(1) \leq t\}, X_{\tau(2)} \cdot I\{\tau(2) \leq t), \ldots; (\pi_s, s \leq t)\}$, where for any process $Z = (Z_t)_{t\geq 0}$ on $\boldsymbol{B}$ we denote $F_t^Z = \sigma\{Z_s; s \leq t\}$ and F^Z — a nondecreasing, right-continuous family of σ-algebras $(F_t^Z)_{t\geq 0}$. Thus Y can be considered not only as a stochastic discrete-time approximation of X, but as a statistic for estimation of X at times $t \geq 0$.

It is clear that, under conditions (1.3) and (1.4), the process X is square-integrable and that we can estimate the values of X_t given the observations $F_t^Y = \sigma\{X_s; s \leq t\}$ at all times $t \geq 0$ as $\hat{X}_t = \boldsymbol{E}(X_t \mid F_t^Y)$. The greater the intensity of the counting process π, the better the approximation and the better could be the estimation. In some applications, the cost of observation is strictly positive, as is the cost of the variance of error of estimation $\varepsilon_t = X_t - \hat{X}_t$. Thus we can formulate the problem of finding an intensity λ of the counting process of the number of observations π that minimizes the objective function calculated using the expected normalized linear function of π and the expected normalized quadratic form of $\varepsilon = (\varepsilon_t)_{t\geq 0}$ (i.e., the linear form of $\gamma = (\gamma_t)_{t\geq 0}$ with $\gamma_t = \boldsymbol{E}(\varepsilon_t^2 | F_t^Y)$). We consider the problem of finding an optimal intensity parameter λ^* for which

$$\varphi(\lambda^*) = \inf_{\lambda > 0} \phi(\lambda), \tag{1.7}$$