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Hybrid wild bootstrap for nonparametric trend estimation in locally stationary time series

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ABSTRACT

Based on consistency and asymptotic normality of a nonparametric kernel trend estimation in the context of locally stationary processes, validity of a hybrid wild bootstrap approach for estimating the distribution of the nonparametric estimator is established. Simulations are presented.

1. Introduction

In time series, analysis, second order stationarity is a major assumption which allows for the development of a powerful asymptotic theory of statistical inference; see for instance (Brockwell, 1991). However, for many time series, their second order characteristic may change over time. Therefore, for a proper asymptotic analysis of statistics based on nonstationary time series, the 'standard' tools of statistical inference developed under stationarity may fail. To develop a capable asymptotic statistical theory that takes into account nonstationarity, one approach is offered by the class of locally stationary processes, c.f. Dahlhaus (2012). Locally stationary processes are stochastic processes which have a time varying dependence structure that fulfills only locally certain second order stationarity constraints. Assuming a linear representation for such processes driven by i.i.d. innovations, a time varying behavior is achieved by allowing the coefficients of this representation to be time dependent. For a detailed overview and some deep examples see Dahlhaus (2012).

For time series stemming from locally stationary processes, many statistical inference procedures developed for stationary processes can be successfully adapted. This includes among others, parameter estimators, nonparametric regression or curve estimators, see Vogt (2012) and von Sachs and MacGibbon (2000). Also, methods to bootstrap such time series have been developed in the literature. For instance a block based bootstrap approach has been considered by Paparoditis and Politis (2002), a frequency domain approach by Sergides and Paparoditis (2008), while a hybrid wild bootstrap approach by Kreiss and Paparoditis (2015). The later approach is the one that will be used in this paper. In particular, we consider the problem of estimating the trend function of a time series stemming from a locally stationary process. For this we derive the asymptotic properties of a nonparametric kernel estimation of the trend function. We then apply the hybrid wild bootstrap procedure introduced by Kreiss and Paparoditis (2015) to infer properties of this estimator and we establish its asymptotic

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validity. Several simulations illustrate the finite sample performance of the bootstrap procedure and a real data example is discussed.

The paper is organized as follows. In Section 2 the main assumptions imposed on the locally stationary process considered are stated and the basic quantities used are defined. Section 3 deals with asymptotic properties of the nonparametric kernel estimator of the trend function. In Section 4 the hybrid wild bootstrap approach is used to infer properties of the nonparametric kernel estimator and the asymptotic validity of this procedure is established. Section 5 investigates the finite sample performance of the bootstrap approach proposed.

2. Locally stationary processes with trend

We assume that we observe a time series $Y_{1,n}, Y_{2,n}, \ldots, Y_{n,n}$,

$$Y_{t,n} = \mu(t/n) + X_{t,n}, \quad t = 1, 2, \ldots, n, \tag{1}$$

where

(a) $\mu : (0, 1] \to \mathbb{R}$ is a *trend function* with $\mu(\cdot) \in \mathcal{H}(\beta, L)$, $\beta \geq 1$ and $\mathcal{H}(\beta, L)$ are the functions within the Hölder-class with parameters $l = \lfloor \beta \rfloor$, $\beta > 0$, $L > 0$, i.e., the l-th derivative of μ satisfies:

$$\left|\mu^{(l)}(x) - \mu^{(l)}(y)\right| \leq L|x - y|^{\beta - l}, \quad \forall y, x \in (0, 1].$$

For more details on the Hölder-class see (Tsybakov, 2009). Furthermore,

(b) $\{X_{t,n}, t = 1, 2, \ldots, n; n \in \mathbb{N}\}$ is a *locally stationary process* satisfying the following assumptions.

$X_{t,n}$ has the linear representation

$$X_{t,n} = \sum_{j=-\infty}^{\infty} \psi_{t,n}(j)\varepsilon_{t-j}, \quad t = 1, \ldots, n \tag{2}$$

with

(i) $\{\varepsilon_t : t \in \mathbb{Z}\}$ are independent, identically distributed (i.i.d.) random variables with $E(\varepsilon_t) = 0$, $E(\varepsilon_t^2) = 1$ and $E\varepsilon_t^4 < \infty$. Let $\kappa_4 = E(\varepsilon_t^4) - 3$.

(ii) $\sup_{t=1,\ldots,n} |\psi_{t,n}(j)| \leq Kl^{-2}(j)\,\forall j \in \mathbb{Z}$, with K a nonnegative constant independent from n and the positive sequence $\{l(j) : j \in \mathbb{Z}\}$ satisfying $\sum_{j=-\infty}^{\infty} |j|l^{-1}(j) < \infty$.

(iii) There exists a function $\psi(\cdot, j) : (0, 1] \to \mathbb{R}$ with

$$\sup_{u\in[0,1]} |\psi(u, j)| \leq \frac{K}{l(j)} \quad \text{and} \quad \sup_{u\in[0,1]} \left|\frac{\mathrm{d}\psi(u, j)}{\mathrm{d}u}\right| \leq \frac{K}{l(j)}, \tag{3}$$

such that

$$\sup_{1\leq t\leq n} |\psi_{t,n}(j) - \psi(t/n, j)| \leq \frac{K}{l(j)} \tag{4}$$

where $\{l(j) : j \in \mathbb{Z}\}$ and K are defined as above.

The above definition allows for a general class of time varying stochastic processes that includes many interesting processes as special cases, like for instance, time varying autoregressive processes; see Dahlhaus (2012) for details. Let $\Psi_{t,n}(\lambda) := \sum_{j=-\infty}^{\infty} \psi_{t,n}(j) \exp(-i\lambda t)$ and define the *time varying spectral density* as $f(u, \lambda) := (2\pi)^{-1}|\Psi(u, \lambda)|^2$ where $\Psi(t/n, \lambda) := \sum_{j=-\infty}^{\infty} \psi(t/n, j) \exp(-i\lambda t)$. It can be shown that under the assumptions made, $\sup_{t,\lambda} |(2\pi)^{-1}\Psi_{t,n}(\lambda)\overline{\Psi_{t,n}(\lambda)} - f(t/n, \lambda)| = O(n^{-1})$; see Dahlhaus (1996b, 2012). Define further the *time varying autocovariance at time point u and lag k* as

$$c(u, k) := \int_{-\pi}^{\pi} f(u, \lambda)e^{-ik\lambda}d\lambda = \sum_{j=-\infty}^{\infty} \psi(u, j)\, \psi(u, k + j)\,. \tag{5}$$

The function $c(t/n, k)$ provides a sufficient approximation of the autocovariance at lag k of $X_{t,n}$, that is, it holds that $\mathrm{Cov}(X_{t,n}, X_{t+k,n}) = c(t/n, k) + O(n^{-1})$. Furthermore, and because of (ii) the time varying autocovariance fulfills $\sum_{k=-\infty}^{\infty} |c(u, k)k| < \infty,\ \forall u \in (0, 1)$.

3. Nonparametric trend estimation

We consider the case where a nonparametric kernel based approach is used to estimate the trend function $\mu(\cdot)$ of the process $\{Y_{t,n}\}$. To elaborate, let $K(\cdot)$ be a kernel of order $l = \lfloor\beta\rfloor$ having support $[-1, 1]$. Furthermore, let K be at least one time continuous differentiable and assume that the following conditions are satisfied: $\sup_u |K(u)| < \infty,\ \sup_u |K^2(u)| < \infty$ and $\sup_u |\mathrm{d}/\mathrm{d}u\, K(u)| < \infty$. The nonparametric kernel trend estimator at time point $u \in (0, 1)$ is then defined as

$$\hat{\mu}(u) = \frac{1}{nh} \sum_{t=1}^{n} K\left(\frac{t/n - u}{h}\right) Y_{t,n}. \tag{6}$$