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A formula of small time expansion for Young SDE driven by fractional Brownian motion

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ABSTRACT

This paper shows an explicit small time expansion formula of expectation of the solution to Young SDEs driven by fractional Brownian motion $H > 1/2$. The expansion coefficients are obtained by using Malliavin calculus for fractional Brownian motion. Furthermore, we show an analytically tractable expansion formula for the expectation of the solution to a general one-dimensional Young SDE driven by fractional Brownian motion and confirm the validity of our small time expansion through numerical experiments.

1. Introduction

The small time behavior of density or expectation of stochastic processes is one of the important issue in statistics, probability theory and related topics. In the case of SDEs driven by the fractional Brownian motion with Hurst index $H > 1/2$, Nualart and Saussereau (2009) and Hu and Nualart (2007) showed the existence of the absolute continuous of the law and the smoothness of the density, respectively, and then Baudoin and Ouyang (2011) and Inahama (forthcoming) have studied Q2 the small time asymptotics of the density.

In this paper, we give an explicit small time expansion of the expectation of the solution to Young SDEs driven by fractional Brownian motion $H > 1/2$. Moreover, we show an analytically tractable expansion for the expectation of the solution to a general one-dimensional Young SDE. Remarkably, our expansion is easily implementable in numerical viewpoint. We perform some numerical experiments using our small time expansion and show the validity.

2. Basic results and notations on stochastic calculus for fractional Brownian motion

In this section, we briefly summarize the basic results and the notations which are useful for the discussions in this paper. See Alós and Nualart (2003), Baudoin (2012) and Nualart (2006) for more details.

Let $(\Omega, \mathcal{F}, P)$ be the canonical probability space associated with the fractional Brownian motion with Hurst index $H > 1/2$. That is, $\Omega = C_0([0, 1])$ is the space of continuous functions on $[0, 1]$ vanishing at 0, $\mathcal{F}$ is the Borel algebra over Ω, P is the unique probability measure on Ω such that the canonical process $B^H = \{B_t^H = (B_t^{H,1}, \dots, B_t^{H,d}),\ t \in [0, 1]\}$ is a fractional Brownian motion with Hurst parameter H. A d-dimensional fractional Brownian motion with Hurst parameter

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$H \in (1/2, 1)$ is a Gaussian process $B^H = (B_t^{H,1}, \dots, B_t^{H,d}), t \geq 0$, where $B^{H,1}, \dots, B^{H,d}$ are d-independent centered Gaussian processes with covariance function $R(t,s) = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H})$. Let $\mathcal{E}$ be the set of step functions on $[0,1]$ and $\mathcal{H}$ be the closure of $\mathcal{E}$ with respect to the scalar product $\langle (1_{[0,t_1]}, \dots, 1_{[0,t_d]}), (1_{[0,s_1]}, \dots, 1_{[0,s_d]}) \rangle_{\mathcal{H}} = \sum_{i=1}^d R_H(t_i, s_i)$. Let us define $B^H(h) = \int_0^1 \langle h_s, dB_s^H \rangle$ for any $h \in \mathcal{H}$ with $E[B^H(h_1)B^H(h_2)] = \langle h_1, h_2 \rangle_{\mathcal{H}}$ for any $h_1, h_2 \in \mathcal{H}$.

A real valued random variable F is said to be cylindrical if it can be written as $F = f\left(B^H(h_1), \dots, B^H(h_m)\right)$, where $h_i \in \mathcal{H}$ and $f : \mathbf{R}^m \to \mathbf{R}$ is a C_b^∞-function. The set of cylindrical random variable is denoted by $\mathcal{P}$. The Malliavin derivative of $F \in \mathcal{P}$ is given by

$$DF = \sum_{i=1}^m \frac{\partial f}{\partial x_i}\left(B^H(h_1), \dots, B^H(h_m)\right) h_i. \tag{1}$$

For $F \in \mathcal{P}$, we set $D^k F = D \cdots DF$. For any $p \geq 1$, the operator D^k is closable from $\mathcal{P}$ into $L^p(\Omega, \mathcal{H}^{\otimes k})$. For any $p \geq 1$ the Sobolev space $\mathbf{D}^{k,p}$ is the closure of the cylindrical random variable with respect to the norm

$$\|F\|_{k,p} = \left(E[F^p] + \sum_{j=1}^k E[\|D^j F\|^p_{\mathcal{H}^{\otimes j}}] \right)^{1/p}. \tag{2}$$

We set the space of smooth functionals $\mathbf{D}^\infty = \cap_{p\geq 1} \cap_{k \geq 1} \mathbf{D}^{k,p}$. Let $F = (F^1, \dots, F^n)$, $F^i \in \mathbf{D}^\infty$, $i = 1, \dots, n$ and define the matrix

$$\sigma_{i,j}^F = \langle DF^i, DF^j \rangle_{\mathcal{H}}, \quad 1 \leq i,j \leq n. \tag{3}$$

The matrix is called the Malliavin covariance matrix of F. We say that $F = (F^1, \dots, F^n)$, $F^i \in \mathbf{D}^\infty$, $i = 1, \dots, n$ is non-degenerate if the Malliavin covariance matrix σ^F is invertible *a.s.* and

$$\|(\det \sigma^F)^{-1}\|_{L^p(\Omega)} < \infty, \quad 1 \leq p < \infty. \tag{4}$$

Moreover, we set the space of the distributions $\mathbf{D}^{-\infty}$ as the dual of $\mathbf{D}^\infty$. The element $S \in \mathbf{D}^{-\infty}$ is called *Watanabe distribution*. For $(S, G) \in \mathbf{D}^{-\infty} \times \mathbf{D}^\infty$, we define the generalized expectation $E[SG] = {}_{\mathbf{D}^{-\infty}}\langle S, G \rangle_{\mathbf{D}^\infty}$ as the coupling. For a Schwartz distribution $T \in \mathcal{S}'(\mathbf{R}^n)$ and the non-degenerate $F \in (\mathbf{D}^\infty)^n$, the composition $T(F) \in \mathbf{D}^{-\infty}$ is well-defined. Especially, if we take the Dirac delta function δ_ξ at $\xi \in \mathbf{R}^n$, $p^F(\xi) = E[\delta_\xi(F)] = {}_{\mathbf{D}^{-\infty}}\langle \delta_\xi(F), 1 \rangle_{\mathbf{D}^\infty}$ is the density of F. The theory and applications of Watanabe distributions are discussed in Watanabe (1987), Ikeda and Watanabe (1989), Yoshida (1992), Malliavin (1997), Kunitomo and Takahashi (2003), Malliavin and Thalmaier (2006), Nualart (2006) and Takahashi and Yamada (2012, submitted for publication, forthcoming) for example.

3. Small time expansion formula

Consider the following multidimensional SDE driven by fractional Brownian motion with Hurst index $H \in (1/2, 1)$:

$$dX_t^x = V_0(X_t^x)dt + \sum_{i=1}^d V_i(X_t^x) dB_t^{H,i}, \quad t \in [0,1], \tag{5}$$

$$X_0^x = x \in \mathbf{R}^n. \tag{6}$$

Here, the stochastic integral is interpreted in the pathwise Young sense. We put the following conditions.

Assumption 1. 1. $V_i, i = 0, \dots, d$ are bounded smooth functions on $\mathbf{R}^n$ with bounded derivatives at any order.
2. $(V_1(x), \dots, V_d(x))$ linearly spans $\mathbf{R}^n$.

By Assumption 1, we are able to see non-degeneracy of Malliavin covariance matrix of X_t^x (Nualart and Saussereau, 2009). We have the following integration by parts formula (see Chronopoulou and Tindel, 2013 for instance).

Proposition 1. *Let $G \in \mathbf{D}^\infty$ and f be any C^∞ bounded function with bounded derivatives. Then for any $\alpha^{(k)} = (\alpha_1, \dots, \alpha_k)$, $k \geq 1$, there exists $H_{\alpha^{(k)}}(X_t^x, G)$ such that*

$$E[\partial_{\alpha^{(k)}} f(X_t^x) G] = E[f(X_t^x) H_{\alpha^{(k)}}(X_t^x, G)], \tag{7}$$

where $\partial_{\alpha^{(k)}} = \frac{\partial^k}{\partial x_{\alpha_1} \cdots \partial x_{\alpha_k}}$. Moreover, $H_{\alpha^{(k)}}(X_t^x, G)$ is recursively given by

$$H_{(l)}(X_t^x, G) = \sum_{i=1}^d \sum_{j=1}^n \left[G \int_0^t Q_s^{lji} dB_s^{H,i} - \alpha_H \int_0^t \int_0^t D_r^i G Q_s^{lji} |r-s|^{2H-2} dr ds \right],$$

$$H_{\alpha^{(k)}}(X_t^x, G) = H_{(\alpha_k)}(X_t^x, H_{(\alpha_{k-1})}(X_t^x, G)), \tag{8}$$

where $Q_s^{lji} = (\sigma^{X_t})^{-1}_{lj} D_s^i X_t^{x,j}$, $l, j \in \{1, \dots, n\}$, $i \in \{1, \dots, d\}$, $\alpha_H = H(2H-1)$ and the integral with respect to B^H is understood in Young sense.

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