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Joint aggregation of random-coefficient AR(1) processes with common innovations

Vytautė Pilipauskaitė [a,b], Donatas Surgailis [a,*]

[a] Vilnius University, Institute of Mathematics and Informatics, Akademijos 4, LT-08663 Vilnius, Lithuania

[b] Université de Nantes, 1, quai de Tourville BP, Nantes Cedex 1, Nantes, 44313, France

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ABSTRACT

We discuss joint temporal and contemporaneous aggregation of N copies of stationary random-coefficient AR(1) processes with common i.i.d. standardized innovations, when N and time scale n increase at different rate. Assuming that the random coefficient a has a density, regularly varying at $a = 1$ with exponent $-1/2 < \beta < 0$, different joint limits of normalized aggregated partial sums are shown to exist when $N^{1/(1+\beta)}/n$ tends to (i) ∞, (ii) 0, (iii) $0 < \mu < \infty$. The paper extends the results in Pilipauskaitė and Surgailis (2014) from the case of idiosyncratic innovations to the case of common innovations.

1. Introduction

Let $X_i := \{X_i(t),\ t \in \mathbb{Z}\}, i = 1, \dots, N$, be stationary random-coefficient AR(1) processes

$$X_i(t) = a_i X_i(t-1) + \varepsilon(t), \quad t \in \mathbb{Z}, \tag{1.1}$$

with *common* standardized i.i.d. innovations $\{\varepsilon(t),\ t \in \mathbb{Z}\}$ and i.i.d. random coefficients $a_i \in (-1, 1), i = 1, \dots, N$, independent of $\{\varepsilon(t),\ t \in \mathbb{Z}\}$. Consider the double sum

$$S_{N,n}(\tau) := \sum_{i=1}^{N} \sum_{t=1}^{[n\tau]} X_i(t), \quad \tau \geq 0, \tag{1.2}$$

representing joint temporal and contemporaneous aggregate of N individual AR(1) evolutions (1.1) at time scale n. We discuss the limit distribution of appropriately normalized double sums $S_{N,n}$ in (1.2) as N, n jointly increase to infinity, possibly at a *different rate*. Throughout this paper, we suppose that the distribution of generic coefficient $a \in (-1, 1)$ in (1.1), or the mixing distribution, satisfies the following two assumptions.

* Corresponding author.
E-mail addresses: vytaute.pilipauskaite@gmail.com (V. Pilipauskaitė), donatas.surgailis@mii.vu.lt (D. Surgailis).

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Assumption A1. There exist $\beta > -1$ and $\epsilon \in (0, 1)$ such that $\mathrm{P}(a \le x)$ is differentiable on $(1-\epsilon, 1)$ with derivative

$$\mathrm{dP}(a \le x)/\mathrm{d}x = (1-x)^{\beta}\psi(x), \quad x \in (1-\epsilon, 1), \tag{1.3}$$

where ψ is bounded on $(1-\epsilon, 1)$ and continuous at $x = 1$ with $\psi_1 := \lim_{x\to 1}\psi(x) > 0$.

Assumption A2. $\mathrm{E}(1+a)^{-1/2} < \infty$.

Assumptions A1 and A2 refer to the behavior of the mixing distribution in the vicinity of $a = 1$ and $a = -1$, respectively (the positive and negative unit roots of generic AR(1) process $X = X_i$ in (1.1)). Because of oscillation of the moving-average coefficients of X when $a < 0$, the behavior of the mixing distribution near $a = -1$ is generally less important for partial sums processes than its behavior near $a = 1$, the crucial role being played by the parameter β in (1.3). Assumption A1 is similar to Zaffaroni (2004), Puplinskaitė and Surgailis (2010), Pilipauskaitė and Surgailis (2014) and other papers, although the 'typical' range of β is different in the aggregation schemes with common and idiosyncratic innovations. The random-coefficient AR(1) process X has finite variance if and only if $\mathrm{E}X^2(t) = \mathrm{E}\sum_{s\le t} a^{2(t-s)} = \mathrm{E}(1-a^2)^{-1} < \infty$, which implies $\beta > 0$ in (1.3). It is well-known that under the condition (1.3) with $0 < \beta < 1$ (and $a \in [0, 1)$ a.s.), X has long memory in the sense that its covariance decays as $\mathrm{cov}(X(0), X(t)) = O(t^{-\beta})$, $t \to \infty$, so that $\sum_{t=0}^{\infty}|\mathrm{cov}(X(0), X(t))| = \infty$. Zaffaroni (2004) and Puplinskaitė and Surgailis (2009) discussed the existence and long memory properties of the limit (in probability) $\mathfrak{X}(t) := \lim_{N\to\infty} N^{-1}\sum_{i=1}^{N} X_i(t)$, $t \in \mathbb{Z}$, of aggregated AR(1) processes X_i in (1.1), written as a moving-average $\mathfrak{X}(t) = \sum_{j=0}^{\infty} g(j)\varepsilon(t-j)$ with (deterministic) coefficients $g(j) := \mathrm{E}[a^j]$, $j \ge 0$. For $-1/2 < \beta < 0$ in (1.3) and under similar condition on the mixing distribution near $a = -1$, the coefficients $g(j) \sim \Gamma(1+\beta)j^{-\beta-1}$, $j \to \infty$ and the (normalized) partial sum process of $\{\mathfrak{X}(t)\}$ tends to a fractional Brownian motion with parameter $H = (1/2) - \beta \in (1/2, 1)$, see Puplinskaitė and Surgailis (2009, Prop. 2 and 4). We recall that Granger and (1980) proposed the scheme of contemporaneous aggregation of heterogeneous random-coefficient AR(1) processes as a possible explanation of the long memory phenomenon in macroeconomic time series. Subsequently, large-scale contemporaneous aggregation of linear and heteroscedastic heterogeneous time series models was studied in Gonçalves and Gouriéroux (1988), Oppenheim and Viano (2004), Zaffaroni (2004), Zaffaroni (2007), Celov et al. (2007), Puplinskaitė and Surgailis (2009, 2010), Pilipauskaitė and Surgailis (2014) Giraitis et al. (2010) and other papers.

Let us describe the main results of present paper. Assume that the mixing density satisfies Assumptions A1 and A2 with $-1/2 < \beta < 0$ and N, n increase simultaneously so as

$$\frac{N^{1/(1+\beta)}}{n} \to \mu \in [0, \infty], \tag{1.4}$$

leading to the three cases (i)–(iii):

$$\text{Case (i)}: \ \mu = \infty, \qquad \text{Case (ii)}: \ \mu = 0, \qquad \text{Case (iii)}: \ 0 < \mu < \infty. \tag{1.5}$$

Our main result is Theorem 2.1 of Section 2 which states that the 'simultaneous limit' of $S_{N,n}(\tau)$ exists in the sense of weak convergence of finite-dimensional distributions, and is different in all three Cases (i)–(iii), namely,

$$N^{-1}n^{\beta-(1/2)}S_{N,n}(\tau) \to_{\mathrm{fdd}} \sigma_\beta B_{(1/2)-\beta}(\tau) \quad \text{in Case (i)}, \tag{1.6}$$

$$N^{-1/(1+\beta)}n^{-1/2}S_{N,n}(\tau) \to_{\mathrm{fdd}} W_\beta B(\tau) \quad \text{in Case (ii)}, \tag{1.7}$$

$$N^{-1/(1+\beta)}n^{-1/2}S_{N,n}(\tau) \to_{\mathrm{fdd}} \mu^{1/2}Z_\beta(\tau/\mu) \quad \text{in Case (iii)}. \tag{1.8}$$

Here, $B_{(1/2)-\beta}$ is a standard fractional Brownian motion with Hurst parameter $H = (1/2) - \beta$, σ_β is a constant defined in Proposition 2.2(ii), $W_\beta > 0$ is a $(1+\beta)$-stable r.v. independent of a standard Brownian motion B, and Z_β is an 'intermediate process' defined as the double stochastic integral

$$Z_\beta(\tau) := \int_{\mathbb{R}\times\mathbb{R}_+}\left\{\int_0^{\tau} \mathrm{e}^{-x(u-s)}\mathbf{1}(s \le u)\mathrm{d}u\right\}\mathrm{d}B(s)\,N(\mathrm{d}x), \quad \tau \ge 0, \tag{1.9}$$

where $N = \{N(\mathrm{d}x),\ x \in \mathbb{R}_+\}$ is a Poisson random measure on $\mathbb{R}_+ := (0, \infty)$ with intensity $\nu(\mathrm{d}x) := \mathrm{E}N(\mathrm{d}x) := \psi_1 x^{\beta}\mathrm{d}x$, independent of standard Brownian motion B. The existence of the process Z_β in (1.9) and its properties are discussed in Section 2. In particular, we show that Z_β can be regarded as a 'bridge' between the limit processes in Cases (i) and (ii), in the sense that Z_β behaves as $B_{(1/2)-\beta}$ at 'small scales' and as $W_\beta B$ at 'large scales'. See Proposition 2.2 for rigorous formulation.

The present paper extends our previous work (Pilipauskaitė and Surgailis, 2014), where a similar problem was discussed for stationary random-coefficient AR(1) processes $Y_i = \{Y_i(t),\ t \in \mathbb{Z}\}$, $i = 1, \ldots, N$ with *independent* (or idiosyncratic) innovations:

$$Y_i(t) = a_i Y_i(t-1) + \varepsilon_i(t), \quad t \in \mathbb{Z}, \tag{1.10}$$

where $\{\varepsilon_i(t),\ t \in \mathbb{Z}\}$ are independent copies of $\{\varepsilon(t), t \in \mathbb{Z}\}$ in (1.1), independent of $a_i \in [0, 1)$, $i = 1, \ldots, N$. Let $\mathcal{S}_{N,n}(\tau) := \sum_{i=1}^{N}\sum_{t=1}^{[n\tau]} Y_i(t)$, $\tau \ge 0$, be the analogue of $S_{N,n}(\tau)$ in (1.2). Under Assumption A1 with $-1 < \beta < 1$ and N, n