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Likelihood ratio order of sample minimum from heterogeneous Weibull random variables

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ABSTRACT

For heterogeneous Weibull samples with a common shape parameter and weakly majorized scale parameters, we study the likelihood ratio order and the stochastic order between minimums of independent and dependent samples, respectively. Also, an application in fabric strength is presented.

1. Introduction

Order statistics and sample spacings are rather useful in reliability theory, life testing, auction theory and other related areas. For example, the kth order statistic of a sample of size n characterizes the lifetime of a $(n-k+1)$-out-of-n system in reliability, and the sample minimum represents the winner's price in the first-price Dutch auction. Ever since (Pledger and Proschan, 1971) first compared order statistics from heterogeneous exponential samples, many researchers followed them to consider order statistics from various samples, for example Weibull samples, proportional hazard samples, gamma samples etc. For more details, one may refer to Navarro and Lai (2007), Zhao et al. (2009), Mao and Hu (2010), Balakrishnan and Zhao (2013) and references therein. As for stochastic comparisons of sample spacings, Pledger and Proschan (1971), Kochar and Korwar (1996), Kochar and Rojo (1996) and Kochar and Xu (2011) studied independent heterogeneous and homogeneous exponential samples, and Torrado and Lillo (2013) considered two independent heterogeneous exponential samples. For more on this line of research, we refer readers to Kochar (2012) and references therein. We will handle stochastic comparisons of sample minimums from independent heterogeneous and homogeneous Weibull samples and those from two independent heterogeneous Weibull samples in this paper. Also this paper will study sample minimums of Weibull random variables with a common Archimedean survival copula based on the log-convex or log-concave generator. Since the sample minimum of nonnegative random variables is just the first sample spacing, all results to be presented in this note also extend the corresponding literature on stochastic comparison of the first sample spacings.

The Weibull distribution $\mathcal{W}(\alpha, \lambda)$ has the probability density $f(x; \alpha, \lambda) = \alpha x^{\alpha-1}\lambda^{\alpha}e^{-(\lambda x)^{\alpha}}$ for $x > 0, \alpha > 0$ and $\lambda > 0$. Denote $X_{1:n}$ the smallest order statistic of $X_1, \ldots, X_n$. For $X_i \sim \mathcal{W}(\alpha, \lambda_i),\ i = 1, \ldots, n$ and $Y_i \sim \mathcal{W}(\alpha, \mu_i),\ i = 1, \ldots, n$, both mutually independent, it is of both theoretical and practical interest to investigate how the majorization of scale parameters

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$\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ and $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$ impacts the distribution of the sample minimums. Khaledi and Kochar (2006, Theorem 2.3) firstly proved that

$$\boldsymbol{\lambda} \overset{\text{m}}{\preceq} \boldsymbol{\mu} \Longrightarrow \begin{cases} Y_{1:n} \le_{\text{hr}} X_{1:n}, & \text{for } \alpha \ge 1, \\ X_{1:n} \le_{\text{hr}} Y_{1:n}, & \text{for } \alpha \le 1, \end{cases} \tag{1.1}$$

where '$\overset{\text{m}}{\preceq}$' and '$\le_{\text{hr}}$' respectively denote the majorization order and the hazard rate order (see Section 2 for their definitions). Lately, Fang and Zhang (2013, Theorem 3.2) proved that

$$\boldsymbol{\lambda} \overset{\text{m}}{\preceq} \boldsymbol{\mu} \Longrightarrow \begin{cases} Y_{1:n} \le_{\text{rh}} X_{1:n}, & \text{for } \alpha \ge 1, \\ X_{1:n} \le_{\text{rh}} Y_{1:n}, & \text{for } \alpha \le 1, \end{cases} \tag{1.2}$$

where '$\le_{\text{rh}}$' denotes the reversed hazard rate order (see Section 2 for the definition). Further suppose mutually independent $Z_i \sim \mathcal{W}(\alpha, \lambda),\ i = 1, \ldots, n$. Fang and Zhang (2013, Theorem 3.1) also showed that

$$\lambda^n = \prod_{k=1}^{n} \lambda_k \Longrightarrow X_{1:n} \le_{\text{rh}} Z_{1:n}, \tag{1.3}$$

$$\lambda^\alpha \ge \frac{1}{n} \sum_{k=1}^{n} \lambda_k^\alpha \Longrightarrow Z_{1:n} \le_{\text{disp}} X_{1:n}, \tag{1.4}$$

where '$\le_{\text{disp}}$' denotes the dispersive order (see Section 2 for the definition). Recently, for $X_i \sim \mathcal{W}(\alpha, \lambda_i),\ i = 1, \ldots, n$ and $Y_i \sim \mathcal{W}(\alpha, \mu_i),\ i = 1, \ldots, n$, both mutually independent, Torrado (in press, Theorem 3.1) further showed that $X_{1:n} \le_{\text{hr}} Y_{1:n}$ is equivalent to $X_{1:n} \le_{\text{lr}} Y_{1:n}$, here '$\le_{\text{lr}}$' denotes the likelihood ratio order (see Section 2 for the definition), and thus Torrado (in press, Theorem 3.2) pointed out that (1.1) and (1.2) can be strengthened to

$$\boldsymbol{\lambda} \overset{\text{m}}{\preceq} \boldsymbol{\mu} \Longrightarrow \begin{cases} Y_{1:n} \le_{\text{lr}} X_{1:n}, & \text{for } \alpha \ge 1, \\ X_{1:n} \le_{\text{lr}} Y_{1:n}, & \text{for } \alpha \le 1. \end{cases} \tag{1.5}$$

This paper is devoted to developing criteria to verify the existence of the likelihood ratio order between sample minimums in the context that the majorization in (1.5) is relaxed to the weak majorizations. Moreover, the dispersive order in (1.4) is also generalized to samples sharing a common Archimedean survival copula.

The rest of this paper is organized as follows: As a preliminary Section 2 recalls related stochastic orders of random variables, majorization of real vectors and two weak versions, and some useful lemmas to be used in the sequel. In Section 3, we obtain the likelihood ratio order between two sample minimums of independent Weibull samples with a common shape parameter and weakly majorized scale parameters. Section 4 obtains sufficient conditions for both the stochastic order and the dispersive order on sample minimums of Weibull samples sharing a common Archimedean survival copula. Finally, for the likelihood ratio order between sample minimums we present in Section 5 an application in the blended fibre strength.

Throughout this note, for convenience, we use the notations $\mathcal{R} = (-\infty, +\infty)$, $\mathcal{R}_+ = [0, +\infty)$ and $\mathcal{R}_{++} = (0, +\infty)$, we also denote $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$ survival functions of distribution functions F and G, respectively, and the terms *increasing* and *decreasing* mean *nondecreasing* and *nonincreasing*, respectively.

2. Preliminaries

For ease of reference, we recall some related concepts and useful lemmas that play a role in developing our main theories in the sequel.

For two random variables X and Y with distribution functions F and G, density functions f and g, respectively, denote F^{-1} and G^{-1} the right continuous inverses[1] of F and G, X is said to be smaller than Y in the (i) *likelihood ratio order* (denoted as $X \le_{\text{lr}} Y$) if $g(t)/f(t)$ increases in t, (ii) *hazard rate order* (denoted as $X \le_{\text{hr}} Y$) if $\bar{G}(t)/\bar{F}(t)$ is increasing in t, (iii) *reversed hazard rate order* (denoted as $X \le_{\text{rh}} Y$) if $G(t)/F(t)$ is increasing in t, (iv) usual *stochastic order* (denoted as $X \le_{\text{st}} Y$) if $\bar{F}(t) \le \bar{G}(t)$ for all t, and (v) *dispersive order* (denoted as $X \le_{\text{disp}} Y$) if $F^{-1}(\beta) - F^{-1}(\alpha) \le G^{-1}(\beta) - G^{-1}(\alpha)$ for all $0 < \alpha \le \beta < 1$. For more on stochastic orders please refer to Shaked and Shanthikumar (2007) and Li and Li (2013).

Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_n)$ be two real vectors, denote $x_{(1)} \le \cdots \le x_{(n)}$ the increasing arrangement of $x_1, \ldots, x_n$. $\boldsymbol{x}$ is said to be

(i) *weakly submajorized* by $\boldsymbol{y}$ (denoted as $\boldsymbol{x} \preceq_{\text{w}} \boldsymbol{y}$) if $\sum_{i=j}^{n} x_{(i)} \le \sum_{i=j}^{n} y_{(i)}$ for all $j = 1, \ldots, n$,

(ii) *weakly supermajorized* by $\boldsymbol{y}$ (denoted as $\boldsymbol{x} \preceq^{\text{w}} \boldsymbol{y}$) if $\sum_{i=1}^{j} x_{(i)} \ge \sum_{i=1}^{j} y_{(i)}$ for all $j = 1, \ldots, n$,

(iii) *majorized* by $\boldsymbol{y}$ (denoted as $\boldsymbol{x} \overset{\text{m}}{\preceq} \boldsymbol{y}$) if $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$ and $\sum_{i=1}^{j} x_{(i)} \ge \sum_{i=1}^{j} y_{(i)}$ for all $j = 1, \ldots, n-1$.

[1] The right continuous inverse of an increasing function h is defined as $h^{-1}(u) = \sup\{x \in \mathcal{R} : h(x) \le u\}$.