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Some Stein-type inequalities for multivariate elliptical distributions and applications

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ABSTRACT

Brown et al. (2006) derive a Stein-type inequality for the multivariate Student's t-distribution. We generalize their result to the family of (multivariate) generalized hyperbolic distributions and derive a lower bound for the variance of a function of a random variable.

1. Introduction

Stein's lemma (Stein, 1973) shows that for a bivariate normally distributed random vector $(X, Y)^T$,

$$\mathrm{Cov}[h(X), Y] = \mathrm{Cov}[X, Y]\mathrm{E}[h^{'}(X)]. \quad (1)$$

Here, $h(x)$ is any differentiable function such that $\mathrm{E}[|h'(X)|]$ exists. When $X = Y$, this equality is also known as Stein's identity. As this result has wide applicability in statistics, insurance and finance, much effort has been devoted to the development of extensions and generalizations. In this regard, note that Diaconis and Zabell (1991) have shown that (1) only holds when $(X, Y)^T$ is bivariate normally distributed so that any generalization must involve some modification of this basic equality.

Further developments include Adcock (2007) who derived a version of Stein's lemma in the case of skew-normal distributions, and Landsman (2006) who generalized Stein's lemma to bivariate elliptical distributions. More precisely, this author showed that if $(X, Y)^T$ is an elliptical random pair,

$$\mathrm{Cov}[h(X), Y] = \mathrm{Cov}[Y, X]\mathrm{E}[h'(X^*)], \quad (2)$$

where X^* denotes a random variable that is associated with X but does not share the same density function, except in the normal case (a feature that is consistent with the aforementioned result of Diaconis and Zabell (1991). In particular, if $(X, Y)^T$ is bivariate t-distributed with m degrees of freedom, then X^* is a t-distributed variable with $m - 2$ degrees of freedom (see

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Landsman, 2006). Interestingly, Brown et al. (2006) show that Stein's identity for normally distributed risks still holds as an *in*equality for t-distributions. More precisely, using the heat equation these authors find that

$$\mathrm{Cov}[h(X), X] \geq \frac{m}{m-1}\mathrm{E}[h'(X)], \tag{3}$$

where X follows a classical Student's t-distribution with m ($m > 2$) degree of freedom and h is a non-decreasing differentiable function.

Note that all mentioned expressions (1)–(3) can be readily generalized to random vectors $\mathbf{X}$ (and $\mathbf{Y}$). In fact, Brown et al. (2006) derived a multidimensional version of the inequality (3) and used it to show that $\mathbf{X}$ is inadmissible for estimating the location vector of the multivariate t-distribution. Our paper is mostly related to theirs. We make the following contributions:

We build on Stein's lemma directly to provide a short proof for the inequality result (3) of Brown et al. (2006) and we generalize it to the family of (multivariate) generalized hyperbolic distribution (which includes the multivariate t-distribution as a special case). In passing, we also provide an elementary proof of Stein's identity (1) as this appears to be missing in the literature.

We also provide two applications. The first application consists in using the newly proposed Stein inequality to prove inadmissibility of $\mathbf{X}$ for estimating the location parameter when $\mathbf{X}$ follows multivariate generalized hyperbolic distributions, and we propose (biased) estimates of the form $\mathbf{X} + \mathbf{H}(\mathbf{X})$ that are dominating. As a second application, we use Stein's inequality to obtain a simple lower bound for the variance of a function of a random variable. This bound is closely related to the standard Chernoff-type bound.

This rest of the paper is organized as follows. We first recall the definition of multivariate elliptical distributions and recall Stein's seminal lemma in Section 2. In Section 3, we extend the stated Stein-inequality (3) to the family of generalized hyperbolic distributions. We show that the results in Brown et al. (2006) are a special case of ours. Section 4 provides the two mentioned applications of the Stein-inequality and Section 5 concludes the paper.

2. Stein's lemma for multivariate elliptical distributions

Definition 1 (*Multivariate Elliptical Distribution*). We say that the random vector $\mathbf{X} = (X_1, \ldots, X_n)^T$ has an elliptical distribution with parameters the $n \times 1$ vector $\boldsymbol{\mu}$ and the $n \times n$ positive definite matrix $\boldsymbol{\Sigma}$ if its characteristic function is given by

$$\mathrm{E}\left[\exp\left(i\mathbf{t}^T\mathbf{X}\right)\right] = \exp\left(i\mathbf{t}^T\boldsymbol{\mu}\right)\,\phi\left(\frac{\mathbf{t}^T\boldsymbol{\Sigma}\mathbf{t}}{2}\right), \quad \mathbf{t}^T = (t_1, t_2, \ldots, t_n), \tag{4}$$

for some scalar function $\phi(t)$ which is called the characteristic generator. We then write $\mathbf{X} \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \phi)$.

A necessary and sufficient condition for the function ϕ to be a characteristic generator of an n-dimensional elliptical distribution is given in Theorem 2.2 of Fang et al. (1990). In this note, we always assume that every mentioned ellipsoidal random vector $\mathbf{X} \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \phi)$ has a density function $f_{\mathbf{X}}(\mathbf{x})$ with vector of expectations $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$ (the latter is obtained by choosing $\phi'(0) = -1$, see Fang et al. (1990)). The density $f_{\mathbf{X}}(\mathbf{x})$ is of the form

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{\sqrt{|\boldsymbol{\Sigma}|}}g_n\left(\frac{(\mathbf{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})}{2}\right), \tag{5}$$

where g_n is a non-negative measurable function such that $\int_{\mathbb{R}^n} g_n(\frac{\mathbf{x}^T\mathbf{x}}{2})dx_1\ldots dx_n = 1$, g_n is usually called the *density generator*. We can now also denote the n-dimensional elliptical distribution that arises from the function g_n (corresponding to the characteristic generating function ϕ) as $\mathbf{X} \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g_n)$. Fang et al. (1990) have shown that if $\mathbf{X} \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g_n)$, then all m-dimensional marginals ($1 \leq m < n$) are also elliptically distributed, with a density generator g_m that can be expressed in terms of g_n as

$$g_m(u) = \frac{(2\pi)^{n/2}}{\Gamma((n-m)/2)}\int_u^\infty (y-u)^{(n-m)/2-1}g_n(y)dy.$$

It is well-known that elliptical family is closed under the affine transform, i.e. when $\mathbf{X} \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\phi})$ then for any $m \times n$ matrix $\mathbf{B}$ with rank m ($m \leq n$) and any $m \times 1$ vector $\mathbf{c}$ it holds that

$$\mathbf{BX} + \mathbf{c} \sim E_m\left(\mathbf{B}\boldsymbol{\mu} + \mathbf{c}, \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}^T, \phi\right), \tag{6}$$

see e.g. Fang et al. (1990). In what follows, let $\mathbf{X}$ be an n-dimensional vector and $\mathbf{Y}$ be m-dimensional. We denote by $\mathrm{Cov}[\mathbf{X}, \mathbf{Y}]$ the $n \times m$ matrix defined as

$$(\mathrm{Cov}[\mathbf{X}, \mathbf{Y}])_{i,j} := \mathrm{Cov}[X_i, Y_j], \quad i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m.$$

When $\mathbf{Y} = \mathbf{X}$ we simply write $\mathrm{Cov}[\mathbf{X}]$ instead of $\mathrm{Cov}[\mathbf{X}, \mathbf{Y}]$. Furthermore, we denote by $\mathrm{Cov}[\mathbf{X}, Y]$ an $n \times 1$ matrix and by $\mathrm{Cov}[Y, \mathbf{X}]$ a $1 \times n$ matrix that are respectively defined as,

$$(\mathrm{Cov}[\mathbf{X}, Y])_i := \mathrm{Cov}[X_i, Y],$$
$$(\mathrm{Cov}[Y, \mathbf{X}])_i := \mathrm{Cov}[Y, X_i], \quad i = 1, 2, \ldots, n. \tag{7}$$