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A matching prior for the shape parameter of the exponential power distribution

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ABSTRACT

We derive a class of matching priors for the shape parameter of the exponential power distribution, which controls the thickness of the density tails. It is shown that a second-order matching prior does not exist in the subclass of the considered priors.

1. Introduction

Consider a simple linear regression model of the form

$$y = X\beta + \varepsilon, \tag{1}$$

where $y = (y_1, \dots, y_n)'$ is an $n \times 1$ vector of response variables, $X = [X_1, \dots, X_p]$ is an $n \times p$ design matrix of predictors, and $\beta = (\beta_1, \dots, \beta_p)'$ is a $p \times 1$ vector of regression coefficients. Here, $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)'$ is an $n \times 1$ vector of independent and identically distributed exponential power (for short, EP) errors with location parameter zero, scale parameter ξ, and shape parameter p. The domains of the model parameter values are given as follows: $(\beta, \xi, p) \in \mathbb{R}^p \times (0, \infty) \times (1, \infty)$.

The probability density function (pdf) of the EP distribution is given by

$$f(y \mid \mu, \xi, p) = \frac{1}{\xi}\left[2p^{1/p}\Gamma(1 + 1/p)\right]^{-1} \exp\left(-\frac{1}{p}\left|\frac{y-\mu}{\xi}\right|^p\right), \tag{2}$$

where $-\infty < y < \infty$, $-\infty < \mu < \infty$, $\xi > 0$, and $p > 1$. One appealing property of the EP distribution is that it can provide both lighter (platykurtic) and heavier (leptokurtic) than the normal density. Specifically, this property is determined by the shape parameter p since kurtosis, a measure of the thickness of the density tails is equal to $\Gamma(1/p)\Gamma(5/p)/[\Gamma(3/p)]^2$. The EP distribution is platykurtic (leptokurtic) for $p > 2$ $(p < 2)$. In addition, it includes several special cases: the Laplace or double

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exponential distribution ($p = 1$), the normal distribution ($p = 2$), and the uniform distribution on the interval $(\mu - \xi, \mu + \xi)$ $(p \to \infty)$.

In a similar way as done by Ferreira and Salazar (2014), we reparameterize the pdf of the EP distribution as

$$f(y \mid \mu, \sigma, p) = \frac{1}{2\sigma} \exp\left(-\left[\Gamma(1+1/p)\right]^p \left|\frac{y-\mu}{\sigma}\right|^p\right),$$

by setting $\sigma = \xi p^{1/p}\Gamma(1+1/p)$. We consider this parameterization since it provides a simplified Fisher information than the one directly derived from the EP density in (2).

The analysis of the EP regression model has recently received attention in the literature due to its flexibility of density tails. The assumption of an EP distribution for error can decrease the influence of outliers and thus increase the robustness of the analysis. For example, Box and Tiao (1973) consider the EP model in the context of robustness of regression models. Achcar and de Araújo Pereira (1999) study the mixture models with use of the EP distribution in the presence of covariates. Salazar et al. (2012) derive three different types of Jeffreys' priors and showed only one of them leads to a proper posterior distribution. More recently, Ferreira and Salazar (2014) consider reference priors of Berger and Bernardo (1992) for all possible parameter orderings and observe that there are only two types of reference priors given by

$$\pi^{r1}(p, \sigma, \beta) \propto \frac{1}{\sigma} p^{-3/2}\left[(1+1/p)\psi'(1+1/p)\right]^{1/2}, \tag{3}$$

and

$$\pi^{r2}(p, \sigma, \beta) \propto \frac{1}{\sigma} p^{-3/2}\left[(1+1/p)\psi'(1+1/p) - 1\right]^{1/2}, \tag{4}$$

where $\psi'(a) = d\{\psi(a)\}/da$ and $\psi(a) = d\{\log\Gamma(a)\}/da$ are the trigamma and digamma functions, respectively. Note that reference prior in (3) is also the independent Jeffreys' prior due to Salazar et al. (2012), who also studied the corresponding propriety of the posterior distribution under the two priors. Note that the two papers by Salazar et al. (2012) and Ferreira and Salazar (2014) only conduct Monte Carlo simulations to investigate the frequentist properties of the Bayesian credible intervals under these priors. They conclude from numerical results that the Bayesian credible intervals have similar frequentist properties with coverage close to nominal for the shape parameter p, whereas there are no theoretical justifications to support their conclusion according to a certain probability matching criterion. This is mainly because such derivation based on probability matching criteria is far from trivial, as commented by Ferreira and Salazar (2014).

Due to the importance of the shape parameter p in the EP regression model, we derive a general class of the first-order matching prior (Datta et al., 2000a) when p is the parameter of interest. This general class includes reference prior in (4) as a special case. Specifically, we study the frequentist properties of the Bayesian credible intervals based on this class of the priors in terms of common probability matching criteria. The matching is accomplished through either (i) posterior quantiles, (ii) distribution functions, or (iii) highest posterior density (HPD) intervals. It is shown that there does not exist any second-order matching prior for p in the subclass of the considered priors. The proposed results are important from both theoretical and practical reasons. From a theoretical perspective, an asymptotic justification is presented for the frequentist properties of Bayesian credible intervals. From a practical perspective, the proposed result provides a guideline to choose appropriate prior distributions, especially from frequentist validity of the Bayesian estimation.

The letter unfolds as follows. In Section 2, we introduce an orthogonal reparameterization of the EP parameters to simplify derivation of the probability matching priors. In Section 3, we consider three common probability matching criteria and derive a general class of the first-order matching priors for the shape parameter p. We show there does not exist any second-order matching prior in the subclass of the considered priors. Some concluding remarks are provided in Section 4.

2. The orthogonal reparameterization

For the given EP regression model in (1), the likelihood function can be written as

$$f(y \mid p, \sigma, \beta) = \frac{1}{2^n\sigma^n} \exp\left(-\left[\Gamma(1+1/p)\right]^p \sum_{i=1}^n \left|\frac{y_i - x_i\beta}{\sigma}\right|^p\right), \tag{5}$$

and the Fisher information matrix $I(p, \sigma, \beta)$ is given by

$$I(p,\sigma,\beta) = \begin{bmatrix} \frac{n}{p^3}\left(1+\frac{1}{p}\right)\psi'\left(1+\frac{1}{p}\right) & -\frac{n}{\sigma p} & 0 \\ -\frac{n}{\sigma p} & \frac{np}{\sigma^2} & 0 \\ 0 & 0 & \frac{1}{\sigma^2}\Gamma\left(\frac{1}{p}\right)\Gamma\left(2-\frac{1}{p}\right)\sum_{i=1}^n x_i x_i' \end{bmatrix}.$$