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# Extremes of random variables observed in renewal times

Bojan Basrak [a], Drago Špoljarić [b,*]

[a] Department of Mathematics, University of Zagreb, Bijenička 30, Zagreb, Croatia
[b] Faculty of Mining, Geology and Petroleum Engineering, University of Zagreb, Pierottijeva 6, Zagreb, Croatia

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ABSTRACT

We use point processes theory to describe the asymptotic distribution of all upper order statistics for observations collected at renewal times. As a corollary, we obtain limiting theorems for corresponding extremal processes.



## 1. Introduction

The maximum of a random number of random variables has been studied for decades. The basic problem is to understand the distribution of

$$M(t) = \max_{i=1,\ldots,\tau(t)} X_i,$$

for an i.i.d. sequence $(X_n)$ and random variables $\tau(t),\ t \geq 0$, which are typically modelled by a renewal process. The earliest references are Lamperti (1961), Berman (1962) and Barndorff-Nielsen (1964), see also Shanthikumar and Sumita (1983) and Anderson (1987) for extensions and applications in engineering. More recently, Meerschaert and Stoev (2009) and Pancheva et al. (2009) studied the convergence of the process $(M(t))$ towards an appropriate extremal process. A general treatment of extremal processes with random sample size can be found in Silvestrov and Teugels (1998, 2004).

From practical perspective, it is frequently important to understand the distribution of all the extreme observations and not merely the maximum. Thus, we aim to explain the limiting behaviour of all large values in the sequence $(X_n)$, which arrive before a given time $t$. To do that, we rely on the theory of point processes. Such an approach seems to be entirely new in this context. It does not only yield more general results, but we believe, it provides a better insight into why previously established results actually hold.

* Corresponding author.
*E-mail addresses:* bbasrak@math.hr (B. Basrak), drago.spoljaric@rgn.hr (D. Špoljarić).

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Throughout $(\tau(t))$ represents the renewal process generated by an i.i.d. sequence of nonnegative random variables $(Y_n)$, i.e.

$$\tau(t) = \inf\{k : Y_1 + \cdots + Y_k > t\}, \quad \text{for } t \geq 0. \tag{1}$$

Moreover, we assume that the distribution of $X_1$ belongs to the maximum domain of attraction (MDA for short) of one of the three extreme value distributions, denoted by $G$. Because of the correspondence between MDA's of Fréchet and Weibull distributions, we discuss only Gumbel and Fréchet MDA's in detail (see subsection 3.3.2 in Embrechts et al., 1997). Recall that $X_1 \in \text{MDA}(G)$ means that for some sequences $(a_n)$ and $(b_n)$

$$nP(X_1 > a_n x + b_n) \to -\log G(x),$$

as $n \to \infty$, for each real $x$ such that $G(x) \in (0, 1)$. This is further equivalent to $(M_n - b_n)/a_n \xrightarrow{d} G$, as $n \to \infty$, where $M_n = \max\{X_i : 1 \leq i \leq n\}$ denotes the partial maxima of the i.i.d. sequence $(X_n)$. Typically, one also assumes that $(\tau(t))$ is independent from sequence of observations $(X_n)$.

The partial maximum of $(X_n)$ governed by $\tau(t)$ is defined as

$$M^\tau(t) = \sup\{X_i : i \leq \tau(t)\}.$$

If the steps of the renewal process have finite mean, that is, if $\mu = EY \in (0, \infty)$, for i.i.d. $X_n$'s, we know that the partial maxima governed by the renewal process behave as if they were observed at deterministic times. In other words

$$\frac{M^\tau(t) - b_{\lfloor t/\mu \rfloor}}{a_{\lfloor t/\mu \rfloor}} \xrightarrow{d} G, \tag{2}$$

as $t \to \infty$. Intuitively, one could say that $M^\tau(t)$ behaves as $M_{\lfloor t/\mu \rfloor}$. Moreover, this holds irrespective of dependence between $(\tau(t))$ and the observations. For the renewal process with infinite mean, but with regularly varying steps, one can still determine the asymptotic distribution of the maximum, see Berman (1962). In such a setting, the convergence of $(M^\tau(t))$ was shown at the level of stochastic processes, see Meerschaert and Stoev (2009) and Pancheva et al. (2009). In the rest of the paper we show how one can move beyond the maxima and extend those results to all upper order statistic in both finite and infinite mean case.

The paper is organized as follows: notation and auxiliary results are introduced in Section 2. In Section 3, we consider the finite mean case in detail, while the problem when interarrival times have infinite mean and are independent of observations will be studied in Section 4.

## 2. Auxiliary point processes

As already mentioned in the introduction, we assume that the distribution of $X_1$ belongs to the MDA(G) where $G$ is Gumbel ($G = \Lambda$) or Fréchet ($G = \Phi_\beta$, for $\beta > 0$) distribution. In particular, there exist functions $a(t)$ and $b(t)$ such that

$$tP(X_1 > a(t)x + b(t)) \to -\log G(x), \tag{3}$$

as $t \to \infty$ (cf. Resnick, 2008). Throughout the article we consider point processes of the form

$$N_t = \sum_{i \geq 1} \delta_{(i/g(t), X_{t,i})}, \tag{4}$$

for a nondecreasing function $g : (0, \infty) \to (0, \infty)$ tending to $+\infty$ as $x \to \infty$, with

$$X_{t,i} = \frac{X_i - b(g(t))}{a(g(t))}, \tag{5}$$

where scaling and centring functions $a(t)$ and $b(t)$ are given in (3).

In the sequel we will allow the function $g$ to depend on the tail of the step size distribution. However, for i.i.d. observations $(X_n)$, it is well known that $X_1 \in \text{MDA}(G)$ is both necessary and sufficient for weak convergence of $(N_t)$. Moreover, the limiting point process, $N$ say, is a Poisson random measure (PRM) with mean measure $\lambda \times \mu_G$ (PRM($\lambda \times \mu_G$) for short), where $\lambda$ denotes the Lebesgue measure and $\mu_G$ represents the measure induced by the nondecreasing function $\log G$. Observe that $N_t$ take value in the space of Radon point measures $M_p([0, \infty) \times \mathbb{E})$, with $\mathbb{E}$ depending on $G$. For instance, in the Gumbel MDA, with $G = \Lambda$, $\mathbb{E} = (-\infty, \infty]$ and the measure $\mu_G$ satisfies $\mu_G(x, \infty] = e^{-x}$ for $x \in \mathbb{R}$. In the Fréchet MDA, with $G = \Phi_\alpha$, $\mathbb{E} = (0, \infty]$ and the measure $\mu_G$ satisfies $\mu_G(x, +\infty] = x^{-\alpha}$ for every $x > 0$. For the Weibull case and the definition of vague topology on the space of point measures $M_p([0, \infty) \times \mathbb{E})$ we refer to Resnick (2008).

Since the distribution of point processes $N_t$ contains the information about all upper order statistics in the sequence $(X_n)$, our plan is to show the convergence of point processes $N_t$ restricted to time intervals determined by a renewal process. An application of the continuous mapping theorem together with Proposition 3.13 in Resnick (2008) yields the following simple result, which plays an important role in the sequel.