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Lower bounds of the Hausdorff dimension for the images of Feller processes

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ABSTRACT

Let $(X_t)_{t\geqslant 0}$ be a Feller process generated by a pseudo-differential operator whose symbol satisfies $\|p(\cdot,\xi)\|_\infty \leqslant c(1+|\xi|^2)$ and $p(\cdot,0)\equiv 0$. We prove that, for a large class of examples, the Hausdorff dimension of the set $\{X_t : t\in E\}$ for any analytic set $E\subset[0,\infty)$ is almost surely bounded below by $\delta_\infty \dim_{\mathrm{H}} E$, where

$$\delta_\infty := \sup\left\{\delta>0 : \lim_{|\xi|\to\infty} \frac{\inf_{z\in\mathbb{R}^d} \operatorname{Re} p(z,\xi)}{|\xi|^\delta} = \infty\right\}.$$

This, along with the upper bound $\beta_\infty \dim_{\mathrm{H}} E$ with

$$\beta_\infty := \inf\left\{\delta>0 : \lim_{|\xi|\to\infty} \frac{\sup_{|\eta|\leqslant|\xi|}\sup_{z\in\mathbb{R}^d} |p(z,\eta)|}{|\xi|^\delta} = 0\right\}$$

established in Böttcher, Schilling and Wang (2014), extends the dimension estimates for Lévy processes of Blumenthal and Getoor (1961) and Millar (1971) to Feller processes.

1. Background and main result

A *Feller process* $(X_t)_{t\geqslant 0}$ with state space $\mathbb{R}^d$ is a strong Markov process such that the associated operator semigroup $(T_t)_{t\geqslant 0}$,

$$T_t u(x) = \mathbb{E}^x(u(X_t)), \qquad u\in C_\infty(\mathbb{R}^d), \quad t\geqslant 0,\ x\in\mathbb{R}^d,$$

($C_\infty(\mathbb{R}^d)$ is the space of continuous functions vanishing at infinity) enjoys the Feller property, i.e. it maps $C_\infty(\mathbb{R}^d)$ into itself. A semigroup is said to be a *Feller semigroup*, if $(T_t)_{t\geqslant 0}$ is a one-parameter semigroup of linear contraction operators $T_t : C_\infty(\mathbb{R}^d)\to C_\infty(\mathbb{R}^d)$ which is strongly continuous: $\lim_{t\to 0}\|T_t u - u\|_\infty = 0$ for any $u\in C_\infty(\mathbb{R}^d)$, and has the sub-Markov property: $0\leqslant T_t u\leqslant 1$ whenever $0\leqslant u\leqslant 1$. The infinitesimal generator $(A, D(A))$ of the semigroup $(T_t)_{t\geqslant 0}$ (or of the process $(X_t)_{t\geqslant 0}$) is given by the strong limit

$$Au := \lim_{t\to 0}\frac{T_t u - u}{t}$$

on the set $D(A)\subset C_\infty(\mathbb{R}^d)$ of all $u\in C_\infty(\mathbb{R}^d)$ for which the above limit exists with respect to the uniform norm.

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Let $C_c^\infty(\mathbb{R}^d)$ be the space of smooth functions with compact support. Under the assumption that the test functions $C_c^\infty(\mathbb{R}^d)$ are contained in $D(A)$, Courrège (1965, Theorem 3.4) proved that the generator A restricted to $C_c^\infty(\mathbb{R}^d)$ is a pseudo-differential operator,

$$Au(x) = -p(x, D)u(x) := -\int e^{i\langle x,\xi\rangle}\, p(x,\xi)\, \hat{u}(\xi)\, d\xi, \quad u \in C_c^\infty(\mathbb{R}^d),$$

with *symbol* $p : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$, where $\hat{u}$ is the Fourier transform of u, i.e. $\hat{u}(x) = (2\pi)^{-d}\int e^{-i\langle x,\xi\rangle} u(\xi)\, d\xi$. The symbol $p(x,\xi)$ is locally bounded in (x,ξ), measurable as a function of x, and for every fixed $x \in \mathbb{R}^d$ it is a continuous negative definite function in the co-variable. This is to say that it enjoys the following Lévy–Khintchine representation,

$$p(x,\xi) = c(x) - i\langle b(x), \xi\rangle + \frac{1}{2}\langle \xi, a(x)\xi\rangle + \int_{z\neq 0}\left(1 - e^{i\langle z,\xi\rangle} + i\langle z,\xi\rangle \mathbb{1}_{\{|z|\leqslant 1\}}\right) \nu(x, dz), \tag{1.1}$$

where $(c(x), b(x), a(x), \nu(x,dz))_{x\in\mathbb{R}^d}$ are the Lévy characteristics: $c(x)$ is a nonnegative measurable function, $b(x) := (b_j(x)) \in \mathbb{R}^d$ is a measurable function, $a(x) := (a_{jk}(x)) \in \mathbb{R}^{d\times d}$ is a nonnegative definite matrix-valued function, and $\nu(x,dz)$ is a nonnegative, σ-finite kernel on $\mathbb{R}^d \times \mathscr{B}(\mathbb{R}^d \setminus \{0\})$ such that for every $x \in \mathbb{R}^d$, $\int_{z\neq 0}(1 \wedge |z|^2)\, \nu(x,dz) < +\infty$. For details and a comprehensive bibliography we refer to the monographs by Jacob (2001) and the survey (Böttcher et al., 2014). Since we will only consider the case where $c \equiv 0$, we will from now on use the Lévy triplet $(b(x), a(x), \nu(x,dz))$.

It is instructive to have a brief look at Lévy processes which are a particular but important subclass of Feller processes. Our standard reference for Lévy processes is the monograph by Sato (1999). A *Lévy process* $(Y_t)_{t\geqslant 0}$ is a stochastically continuous random process with stationary and independent increments. The characteristic exponent (or symbol) $\psi : \mathbb{R}^d \to \mathbb{C}$ of a Lévy process is a continuous negative definite function, i.e. it is given by a Lévy–Khintchine formula of the form (1.1) with characteristics $(b, a, \nu(dz))$ which do not depend on x.

The notion of Hausdorff dimension is very useful in order to characterize the irregularity of stochastic processes. The Hausdorff dimension of the image sets of a Lévy process has been extensively studied, see Blumenthal and Getoor (1961); Pruitt (1970); Millar (1971) and also the survey papers Taylor (1986); Xiao (2004) for details. Recall that the *Hausdorff dimension* of a set $A \subset \mathbb{R}^d$ is the unique number λ, where the λ-dimensional *Hausdorff measure* $\mathscr{H}^\lambda(A)$, defined by

$$\mathscr{H}^\lambda(A) = \sup_{\varepsilon>0}\inf\left\{\sum_{n=1}^\infty (\operatorname{diam} A_n)^\lambda : A_n \text{ Borel}, \ \bigcup_{n=1}^\infty A_n \supset A \text{ and } \ \operatorname{diam} A_n \leqslant \varepsilon\right\},$$

changes from $+\infty$ to a finite value.

For the study of the Hausdorff dimension for the sample paths of Lévy processes, various indices were introduced in Blumenthal and Getoor (1961, Sections 2, 3 and 5):

$$\beta'' = \sup\left\{\delta > 0 : \lim_{|\xi|\to\infty} \frac{\operatorname{Re}\psi(\xi)}{|\xi|^\delta} = \infty\right\},$$

$$\beta = \inf\left\{\delta > 0 : \lim_{|\xi|\to\infty} \frac{|\psi(\xi)|}{|\xi|^\delta} = 0\right\}.$$

The results for the Hausdorff dimension of the image sets of Lévy processes can be summarized as follows, see Blumenthal and Getoor (1961); Pruitt (1970); Millar (1971):

Remark 1.1. Let $(Y_t)_{t\geqslant 0}$ be a d-dimensional Lévy process with indices β'' and β given above. For every analytic set $E \subset [0, 1]$ we have, almost surely

$$\min\{d, \beta''\dim_{\mathrm{H}} E\} \leqslant \dim_{\mathrm{H}} Y(E) \leqslant \min\{d, \beta\dim_{\mathrm{H}} E\}.$$

Now, we turn to the Hausdorff dimension for the image of a Feller processes. Throughout we will make the following assumptions on the symbol $p(x,\xi)$:

$$\|p(\cdot,\xi)\|_\infty \leqslant c(1 + |\xi|^2) \quad \text{and} \quad p(\cdot, 0) \equiv 0. \tag{1.2}$$

The first condition means that the generator has only bounded 'coefficients', see, e.g. Schilling (1998b, Lemma 2.1) or Schilling and Schnurr (2010, Lemma 6.2); the second condition implies then (if the first condition is satisfied) the Feller process is conservative in the sense that the life time of the process is almost surely infinite, see Schilling (1998a, Theorem 5.2).

We first recall the following upper bound for the Hausdorff dimension for the image of a Feller processes, which partly extends Remark 1.1.

Theorem 1.2 (*Böttcher et al., 2014, Theorem 5.15*)**.** *Let $(X_t)_{t\geqslant 0}$ be a Feller process with the generator $(A, D(A))$ such that $C_c^\infty(\mathbb{R}^d) \subset D(A)$, i.e. $A|_{C_c^\infty(\mathbb{R}^d)} = -p(\cdot, D)$ is a pseudo-differential operator with symbol $p(x,\xi)$. Assume that the symbol satisfies (1.2). Then, for every bounded analytic set $E \subset [0,\infty)$,*

$$\dim_{\mathrm{H}} X(E) \leqslant \min\{d, \beta_\infty \dim_{\mathrm{H}} E\} \tag{1.3}$$