



Almost sure behavior of functionals of stationary Gibbs point processes

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ABSTRACT

This paper is concerned with the almost sure control of functionals of stationary Gibbs point processes. We apply Kahane–Khintchine's inequality to derive an almost sure control of various functionals under very mild assumption on the spatial point process $\mathbf{X}$. In particular, if $\mathbf{X}$ is a locally stable Gibbs point process with finite range observed in $[-n, n]^d$, we obtain the bound $N_{[-n, n]^d}(\mathbf{X})/(2n)^d = \rho + \mathcal{O}_{a.s.}(n^{-d/2} \log n^{3/2})$ as $n \rightarrow \infty$, where $N_W(\mathbf{X})$ is the number of points of $\mathbf{X} \cap W$ for $W \subset \mathbb{R}^d$ and where ρ is the intensity parameter of $\mathbf{X}$.

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1. Introduction

Spatial point patterns are datasets containing the random locations of some event of interest. These datasets arise in many scientific fields such as biology, epidemiology, seismology, and hydrology. Spatial point processes are the stochastic models generating such data. We refer to [Stoyan et al. \(1995\)](#) or [Møller and Waagepetersen \(2004\)](#) for an overview on spatial point processes. These references cover practical as well as theoretical aspects. A point process $\mathbf{X}$ in $\mathbb{R}^d$ is a locally finite random subset of $\mathbb{R}^d$ meaning that the restriction to any bounded Borel set is finite. The point process $\mathbf{X}$ takes values in Ω consisting in all locally finite subsets of $\mathbb{R}^d$. Thus the distribution of $\mathbf{X}$ is a probability measure on an appropriate σ -algebra consisting of subsets of Ω . If the distribution of $\mathbf{X}$ is invariant by translation, we say that $\mathbf{X}$ is stationary. The reference model is the spatial Poisson point process which models random locations of points without any interaction between points. Several classes of alternative models allow to introduce dependence. The class of Gibbs (or Markov) point processes constitutes one of the main alternatives. In a bounded domain, the distribution of a Gibbs point process is defined through a density with respect to the distribution of a Poisson point process with fixed intensity (usually 1). In an infinite volume domain, Gibbs point processes are defined via conditional densities and most of them can be entirely defined via the Papangelou conditional intensity denoted by λ and defined from $\mathbb{R}^d \times \Omega$ to $\mathbb{R}^+$. For $u \in \mathbb{R}^d$, $\lambda(u, \mathbf{X}) du$ can be interpreted heuristically as the conditional probability of observing a point in a ball with volume du around u given the rest of the configuration is $\mathbf{X}$. Another way of characterizing a Gibbs point process is through the Georgii–Nguyen–Zessin formula (see [Georgii, 1976](#); [Papangelou, 2009](#); [Nguyen and Zessin, 1979](#); [Zessin, 2009](#)) which states that for any $h : \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}$ (such that the following expectations are finite)

$$\mathbb{E} \sum_{u \in \mathbf{X}} h(u, \mathbf{X} \setminus u) = \mathbb{E} \int_{\mathbb{R}^d} h(u, \mathbf{X}) \lambda(u, \mathbf{X}) du. \quad (1.1)$$

Let L_h and R_h denote the random variables from the left and the right-hand side of (1.1). We call a functional of a Gibbs point process, a functional of the form $L_h - \mathbb{E} L_h$ (which equals $L_h - \mathbb{E} R_h$), $L_h - R_h$ or $R_h - \mathbb{E} R_h$. Such functionals are of special interest.

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For instance when $\mathbf{X}$ is stationary and $h(u, \mathbf{x}) = \mathbf{1}(u \in W)$, where W is a subset of $\mathbb{R}^d$ with volume $|W| < \infty$, then $L_h - E L_h$ is the difference (up to a normalization) between $N_W(\mathbf{X})/|W|$, where $N_W(\mathbf{X})$ is the random variable counting the number of points of $\mathbf{X}$ in W , and $E \lambda(0, \mathbf{X}) = \rho$, which is nothing else than the intensity parameter of $\mathbf{X}$ i.e. the mean number of points per unit volume (see e.g. Møller and Waagepetersen, 2004). Functionals of Gibbs point processes of the form $L_h - R_h$ have been considered in Baddeley et al. (2005, 2008) and in Coeurjolly and Lavancier (2013). Correctly renormalized, such functionals have been shown to converge almost surely to zero as the window of observation expands to $\mathbb{R}^d$. Coeurjolly and Lavancier (2013) have also established an asymptotic normality result. The goal of this paper is to go further in the study of such functionals by stating an almost sure rate of convergence. The main tool is the application of Kahane–Khintchine’s inequality, obtained by El Machkouri (2002a,b), to the context of Gibbs point processes.

2. Background and main result

2.1. Context

Let $\lambda : \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}^+ \setminus \{0\}$ be a positive function. We say that

[TI] λ is translation invariant if $\lambda(u, \mathbf{x}) = \lambda(0, \tau_{-u}\mathbf{x})$ for any $u \in \mathbb{R}^d$ and $\mathbf{x} \in \Omega$ where τ_{-u} denotes the translation of $\mathbf{x}$ by the vector $-u$.

[FR] λ is of finite range if there exists $0 < R < \infty$ such that for any $u \in \mathbb{R}^d$ and $\mathbf{x} \in \Omega$, $\lambda(u, \mathbf{x}) = \lambda(u, \mathbf{x} \cap B(u, R))$ where $B(u, r)$ denotes the Euclidean ball centered at $u \in \mathbb{R}^d$ with radius r .

[LS] λ is locally stable if there exists $0 < L < \infty$ such that for any $u \in \mathbb{R}^d$ and $\mathbf{x} \in \Omega$, $\lambda(u, \mathbf{x}) \leq L$.

For instance, the Strauss point process defined by $\lambda(u, \mathbf{x}) = \beta \gamma^{N_{B(u, R)}(\mathbf{x})}$ for $\beta > 0$ and $\gamma \in (0, 1]$, or the area interaction process defined by $\lambda(u, \mathbf{x}) = \beta \gamma^{-U(u, \mathbf{x})}$ with $\beta > 0$ and $\gamma > 0$ where $U(u, \mathbf{x}) = A(\mathbf{x} \cup u) - A(\mathbf{x})$ and $A(\mathbf{x}) = \bigcup_{u \in \mathbf{x}} |B(u, R/2)|$, satisfy [TI], [FR] and [LS]. Gibbs point processes satisfying the three above properties constitute the main class of examples (see Møller and Waagepetersen, 2004 for many more examples). In this section we consider positive functions λ , i.e. every configuration of points is admissible. Hard-core point processes, i.e. point processes for which two points at distance lower than δ are forbidden are discussed in Section 4. The existence of Gibbs point processes with prescribed Papangelou conditional intensity is a very difficult question, see Dereudre et al. (2012) for a recent discussion. Under the setting of this paper, we use the results of Bertin et al. (1999) and Georgii and Küneth (1997) to derive the following useful lemma.

Lemma 2.1. *Under the assumptions [TI], [FR] and [LS], there exists a stationary Gibbs measure, i.e. there exists a stationary Gibbs point process with Papangelou conditional intensity λ . In addition, for any $W \subset \mathbb{R}^d$ with $|W| < \infty$, we have $E \exp(cN_W(\mathbf{X})) < \infty$ for any $c > 0$.*

Under the assumptions of Lemma 2.1, the Gibbs measure is not necessarily unique. In the non-uniqueness case, we say that the phase transition occurs and the set of Gibbs measures is a Choquet simplex meaning that any Gibbs measure is a mixture of extremal ergodic Gibbs measures. If the Gibbs measure is unique, it is necessarily ergodic, see e.g. Georgii (1976). Additional comments and refinements of Lemma 2.1 are presented in Section 4.

2.2. Main result

The aim of this paper is to control the almost sure behavior of two kinds of functionals. Let $W \subset \mathbb{R}^d$ and $h : \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}$

$$F_W(\mathbf{X}, h) = \sum_{u \in \mathbf{X} \cap W} h(u, \mathbf{X} \setminus u) - \int_W E(h(u, \mathbf{X}) \lambda(u, \mathbf{X})) du$$

$$G_W(\mathbf{X}, h) = \sum_{u \in \mathbf{X} \cap W} h(u, \mathbf{X} \setminus u) - \int_W h(u, \mathbf{X}) \lambda(u, \mathbf{X}) du.$$

From the Georgii–Nguyen–Zessin formula (1.1), we have $E G_W(\mathbf{X}, h) = E F_W(\mathbf{X}, h) = 0$ for any bounded domain $W \subset \mathbb{R}^d$. Without additional assumption on $\mathbf{X}$, Coeurjolly and Lavancier (2013) also proved that for any bounded domain W

$$E(G_W(\mathbf{X}, h) | \mathbf{X}_{W^c}) = 0 \quad (2.1)$$

almost surely. We present now a few examples of functionals F or G which are of special interest.

- (a) The functional G_W corresponds to the concept of innovations of a spatial point process, introduced by Baddeley et al. (2005, 2008). This functional plays a fundamental role in parametric estimation procedures as well as in procedures to judge the quality of a model. For instance, if $\mathbf{X}$ is the Strauss point process or the area interaction process, then the score function of the pseudo-likelihood which is one of the most well-known methods to estimate a parametric model

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