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Laws of the iterated logarithm of Chover-type for operator stable Lévy processes

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ABSTRACT

Let $X = \{X(t), t \in \mathbb{R}_+\}$ be an operator stable Lévy process in $\mathbb{R}^d$ with exponent E, where E is an invertible linear operator on $\mathbb{R}^d$. Integral tests for sample paths of operator stable Lévy process X are given. Laws of the iterated logarithm of Chover-type are derived from them as corollaries. Our results give information about the maximal growth rate of sample paths of X in terms of the real parts of the eigenvalues of E.

1. Introduction

Let $X = \{X(t), t \in \mathbb{R}_+\}$ be a Lévy process in $\mathbb{R}^d$, that is, X has stationary and independent increments, $X(0) = 0$ a.s. and such that $t \mapsto X(t)$ is continuous in probability. The finite-dimensional distributions of a Lévy process X are completely determined by the characteristic function of $X(t)$ given by

$$\mathbb{E}[e^{i\langle \xi, X(t)\rangle}] = e^{-t\psi(\xi)},$$

where, by the Lévy–Khintchine formula,

$$\psi(\xi) = i\langle a, \xi\rangle + \frac{1}{2}\langle \xi, \Sigma\xi'\rangle + \int_{\mathbb{R}^d}\Big[e^{i\langle x,\xi\rangle} - 1 - \frac{i\langle x, \xi\rangle}{1+\|x\|^2}\Big]\phi(dx), \quad \forall \xi \in \mathbb{R}^d \tag{1.1}$$

and, $a \in \mathbb{R}^d$ is fixed, Σ is a non-negative definite, symmetric, $(d \times d)$ matrix, and ϕ is a Borel measure on $\mathbb{R}^d \setminus \{0\}$ that satisfies

$$\int_{\mathbb{R}^d} \frac{\|x\|^2}{1+\|x\|^2}\phi(dx) < \infty.$$

The function ψ is called the Lévy exponent of X, and ϕ is the corresponding *Lévy measure*. We refer to Bertoin (1996) and Sato (1999) for general theory of Lévy processes.

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There has been considerable interest in studying sample path properties of Lévy processes. Many authors have investigated fractal nature of various random sets generated by Lévy processes. See the survey papers of Taylor (1986) and Xiao (2004) and the references therein for more information. For a stable Lévy process X in $\mathbb{R}^d$ with index $\alpha \in (0, 2)$, many of the results on the sample paths of X can be formulated nicely in terms of α and d. For strictly α-stable processes $X = \{X(t), t \in \mathbb{R}_+\}$ on $\mathbb{R}^1$ with $0 < \alpha < 2$, Chover (1966) showed that the LIL of Chover-type holds. Its integral test was first given by Khintchine (1938), and it implies that

$$\limsup_{t\to\infty} \frac{|X(t)|}{(th_t)^{1/\alpha}} = 0 \quad \text{a.s. or } = \infty \text{ a.s.}$$

according as

$$\int_1^\infty \frac{dt}{th_t} < \infty \text{ or } = \infty,$$

if h_t is increasing and $\lim_{t\to\infty} h_t = \infty$. This fact was shown by Yamamuro's result (2003) for Lévy processes in the case where $\phi((-\infty, 0)) > 0$ and $\phi((0, \infty)) > 0$, and a similar result is reported by Fristedt (1974) for $\phi((0, \infty)) > 0$. Watanabe (1996) investigated the integral test for self-similar processes with independent increments in $\mathbb{R}^1$. Yamamuro (2005) investigated the integral test for a stable Lévy process X in $\mathbb{R}^d$ with index $\alpha \in (0, 2)$.

The purpose of this paper is to prove integral tests for operator stable Lévy processes without Gaussian component. They will include the related results mentioned above as special cases. Laws of the iterated logarithm (LIL) of Chover-type are derived from them as corollaries. Our results give information about the maximal growth rate of sample paths of operator stable Lévy processes.

We use the notations $f_t \sim g_t$ if $\lim f_t/g_t = 1$ and $f_t \asymp g_t$ if there exist constants $c_1, c_2 > 0$ such that $c_1 \le \liminf f_t/g_t \le \limsup f_t/g_t \le c_2$. All constants c appearing in this paper (with or without subscript) are positive and may not necessarily be the same in each occurrence. More specific constants will be denoted by $c_0, c_1, c_2, \ldots$.

2. Main results

A Lévy process $X = \{X(t), t \in \mathbb{R}_+\}$ in $\mathbb{R}^d$ is called *operator stable* if the distribution ν of $X(1)$ is full [i.e., not supported on any $(d-1)$-dimensional hyperplane] and ν is *strictly operator stable*, i.e., there exists a linear operator E on $\mathbb{R}^d$ such that

$$\nu^t = t^E \nu \quad \text{for all } t > 0, \tag{2.1}$$

where ν^t denotes the t-fold convolution power of the infinitely divisible law ν and $t^E\nu(dx) = \nu(t^{-E}dx)$ is the image measure of ν under the linear operator t^E, which is defined by

$$t^E = \sum_{n=0}^{\infty} \frac{(\log t)^n}{n!} E^n, \quad t > 0.$$

The linear operator E is called a *stability exponent* of X. The set of all possible exponents of an operator stable law is characterized in Theorem 7.2.11 of Meerschaert and Scheffler (2001).

On the other hand, a stochastic process $X = \{X(t), t \in \mathbb{R}_+\}$ with values in $\mathbb{R}^d$ is said to be *operator self-similar* if there exists a linear operator E on $\mathbb{R}^d$ such that for every $\lambda > 0$,

$$\{X(\lambda t), t \ge 0\} \stackrel{\mathrm{d}}{=} \{\lambda^E X(t), t \ge 0\}, \tag{2.2}$$

where $X \stackrel{\mathrm{d}}{=} Y$ denotes that the two processes X and Y have the same finite-dimensional distributions. Here the linear operator E is called a *self-similarity exponent* of X.

Hudson and Mason (1982) proved that if X is a Lévy process in $\mathbb{R}^d$ such that the distribution of $X(1)$ is full, then X is operator self-similar if and only if $X(1)$ is strictly operator stable. In this case, every stability exponent E of X is also a self-similarity exponent of X. Hence, from now on, we will simply refer to E as an exponent of X.

In this paper we restrict our study to an operator stable Lévy process without Gaussian component. To describe the characteristic function of $X(1)$, we let

$$S_E = \{x \in \mathbb{R}^d : \|x\|_E = 1\},$$

where

$$\|x\|_E = \int_0^\infty \|e^{-tE}x\| dt = \int_0^1 \|s^E x\| s^{-1} ds$$

is a norm on $\mathbb{R}^d$, that is, S_E is a unit sphere in $\mathbb{R}^d$ with respect to this norm. Then, the Lévy measure ϕ of $X(1)$ in (1.1) (Jurek and Mason, 1993, Proposition 4.3.4) is

$$\phi(A) = \int_{S_E} \int_0^\infty I_A(s^E x) s^{-2} ds\, m(dx) \tag{2.3}$$

for all $A \subset \mathbb{R}^d \setminus \{0\}$, where m is the finite measure on $\mathcal{B}(S_E)$ the class of Borel subsets of S_E, given by

$$m(F) = \phi(\{s^E x : x \in F, s \in [1, \infty)\}).$$