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Quasi-diagonal exponent symmetry model for square contingency tables with ordered categories

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ABSTRACT

For square contingency tables, we propose a quasi-symmetry model with an exponential form along subdiagonal and give the theorem that Tomizawa's (1992) diagonal exponent symmetry model holds if and only if the proposed model and marginal means equality model hold with the orthogonality of test statistics.

1. Introduction

For an $R \times R$ square contingency table with the same row and column classifications, let p_{ij} denote the probability that an observation will fall in the ith row and jth column of the table ($i = 1, \ldots, R;\ j = 1, \ldots, R$). The symmetry (S) model is defined by

$$p_{ij} = \psi_{ij} \quad (i = 1, \ldots, R;\ j = 1, \ldots, R),$$

where $\psi_{ij} = \psi_{ji}$ (Bishop et al., 1975, p. 282). Caussinus (1965) considered the quasi-symmetry (QS) model, defined by

$$p_{ij} = \alpha_i \beta_j \psi_{ij} \quad (i = 1, \ldots, R;\ j = 1, \ldots, R),$$

where $\psi_{ij} = \psi_{ji}$. A special case of this model with $\{\alpha_i = \beta_i\}$ is the S model. The marginal homogeneity (MH) model is defined by

$$p_{i\cdot} = p_{\cdot i} \quad (i = 1, \ldots, R),$$

where $p_{i\cdot} = \sum_{t=1}^{R} p_{it}$ and $p_{\cdot i} = \sum_{s=1}^{R} p_{si}$ (Stuart, 1955). Caussinus (1965) gave the theorem that the S model holds if and only if both the QS and MH models hold.

Tomizawa (1992) considered the diagonal exponent symmetry (DES) model, defined by

$$p_{ij} = p_{ji} = \mu_{|j-i|} \gamma^{i-1} \quad (i < j).$$

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By putting $\gamma^{\frac{1}{2}} = \delta$ and $\mu_{|j-i|}\gamma^{-1-\frac{1}{2}|j-i|} = d_{|j-i|}$, this model may be expressed as

$$p_{ij} = \begin{cases} \delta^{i+j} d_{|j-i|} & (i \neq j), \\ \psi_{ii} & (i = j). \end{cases}$$

The DES model states that, in addition to the structure of the S model, $p_{i+1,j+1}$ $(i \neq j)$ is γ times higher than p_{ij}. Namely, for fixed distance k $(k = 1, \dots, R-2)$ from the main diagonal of the table, $p_{i,i+k}$ (or $p_{i+k,i}$) increases (decreases) exponentially along every subdiagonal of the table as the value of i increases $(i = 1, \dots, R-k)$ with the structure of the S model. We are now interested in proposing a new model such that, in addition to the structure of the QS model (instead of the S model), the expected frequency has an exponential form along every subdiagonal of the table. Also we are interested in considering a decomposition of the DES model using the proposed model (as the decomposition of the S model into the QS and MH models).

Section 2 proposes a new model. Section 3 gives the decomposition of the DES model. Section 4 shows the orthogonality of the test statistics for decomposed models.

2. Quasi-diagonal exponent symmetry model

Consider a model defined by

$$p_{ij} = \begin{cases} \alpha^i \beta^j d_{|j-i|} & (i \neq j), \\ \psi_{ii} & (i = j). \end{cases}$$

We shall refer to this model as the quasi-diagonal exponent symmetry (QDES) model. The QDES model is a special case of the QS model and a special case of the paired diagonals model (Bishop et al., 1975, p. 322). A special case of the QDES model obtained by putting $\alpha = \beta$ is the DES model. The QDES model states that, in addition to the structure of the QS model, $p_{i+1,j+1}$ for $i \neq j$ is $\alpha\beta$ times higher than p_{ij}; in other words, for fixed distance k $(k = 1, \dots, R-2)$ from the main diagonal of the table, $p_{i,i+k}$ (or $p_{i+k,i}$) increases (decreases) exponentially as the value of i increases $(i = 1, \dots, R-k)$.

Denote the odds ratio for rows i and s $(> i)$, and columns j and t $(> j)$ by $\theta_{(i<s;\, j<t)}$; thus $\theta_{(i<s;\, j<t)} = (p_{ij}p_{st})/(p_{it}p_{sj})$. Under the QDES model, we obtain

$$\theta_{(i<s;\, j<t)} = \theta_{(j<t;\, i<s)} \quad (i < s;\ j < t),$$

and

$$\theta_{(i<s;\, j<t)} = \theta_{(i+c<s+c;\, j+c<t+c)} \quad (i < s < j < t;\ c = 1, \dots, R-t).$$

Thus under this model we see that (i) the odds ratios are symmetry (being the structure of the QS model), and (ii) the parallel odds ratios for the main diagonal of the table are the same.

3. Decomposition

Let X and Y denote the row and column variables, respectively. Refer to the model of equality of marginal means, i.e., $\mathrm{E}(X) = \mathrm{E}(Y)$, as the ME model. We obtain the following theorem.

Theorem 1. *The DES model holds if and only if the QDES and ME models hold.*

Proof. If the DES model holds, then the QDES and ME models hold. Assuming that both the QDES and ME models hold, then we shall show that the DES model holds. The ME model can be expressed as

$$\sum_{k=1}^{R-1} G_{1(k)} = \sum_{k=1}^{R-1} G_{2(k)},$$

where

$$G_{1(k)} = \sum_{s=1}^{k} \sum_{t=k+1}^{R} p_{st}, \quad \text{and} \quad G_{2(k)} = \sum_{s=k+1}^{R} \sum_{t=1}^{k} p_{st}.$$

We see

$$\begin{aligned} \sum_{k=1}^{R-1} G_{1(k)} - \sum_{k=1}^{R-1} G_{2(k)} &= \sum_{k=1}^{R-1} \sum_{s=1}^{R-k} k p_{s,s+k} - \sum_{k=1}^{R-1} \sum_{s=1}^{R-k} k p_{s+k,s} \\ &= \sum_{k=1}^{R-1} \sum_{s=1}^{R-k} k \alpha^s \beta^{s+k} d_k - \sum_{k=1}^{R-1} \sum_{s=1}^{R-k} k \alpha^{s+k} \beta^s d_k \end{aligned}$$