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Random walks in I.I.D. random environment on Cayley trees

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ABSTRACT

We consider the random walk in an *i.i.d.* random environment on the infinite d-regular tree for $d \geq 3$. We consider the tree as a Cayley graph of the free product of finitely many copies of $\mathbb{Z}$ and $\mathbb{Z}_2$ and define the i.i.d. environment as invariant under the action of this group. Under a mild non-degeneracy assumption we show that the walk is always transient.

1. Introduction

In this short note we consider a random walk in random environment (RWRE) model on a regular tree with degree $d \geq 3$, where the environment at the vertices is *independent* and is also "*identically distributed*" (i.i.d.). We make this notion of *i.i.d. environment* rigorous by first defining a translation invariant model on a group G which is a free product of finitely many groups, $G_1, G_2, \ldots, G_k$ and $H_1, H_2, \ldots, H_r$, where each $G_i \cong \mathbb{Z}$ and each $H_j \cong \mathbb{Z}_2$ with $d = 2k + r$. Observing the fact that the Cayley graph of G is a regular tree with degree d, we transfer back the model on the d-regular tree we started with. We prove that under a mild non-degeneracy assumption such a walk is always transient.

1.1. Basic setup

Cayley graph: Let G be a group defined above, that is, G is a free product of $k + r \geq 2$ groups, namely $G_1, G_2, \ldots, G_k$ with $k \geq 0$ and $H_1, H_2, \ldots, H_r$ with $r \geq 0$, where each $G_i \cong \mathbb{Z}$ and each $H_j \cong \mathbb{Z}_2$ and $d = 2k + r \geq 3$. Suppose $G_i = \langle a_i \rangle$ for $1 \leq i \leq k$ and $H_j = \langle b_j \rangle$ where $b_j^2 = e$ for $1 \leq j \leq r$. Here by $\langle a \rangle$ we mean the group generated by a single element a. Let $S := \{a_1, a_2, \ldots, a_k\} \cup \left\{a_1^{-1}, a_2^{-1}, \ldots, a_k^{-1}\right\} \cup \{b_1, b_2, \ldots, b_r\}$ be a generating set for G. We note that S is a symmetric set, that is, $s \in S \iff s^{-1} \in S$.

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We now define a graph $\bar{G}$ with vertex set G and edge set $E := \left\{\{x, y\} \middle| yx^{-1} \in S\right\}$. Such a graph $\bar{G}$ is called a *(left) Cayley Graph* of G with respect to the generating set S. Since G is a free product of groups which are isomorphic to either $\mathbb{Z}$ or $\mathbb{Z}_2$, it is easy to see that $\bar{G}$ is a graph with no cycles and is regular with degree d, thus it is isomorphic to the d-regular infinite tree which we will denote by $\mathbb{T}_d$. We will abuse the terminology a bit and will write $\mathbb{T}_d$ for the Cayley graph of G. We will consider the identity element e of G as the root of $\mathbb{T}_d$. We will write $N(x)$ for the set of all neighbors of a vertex $x \in \mathbb{T}_d$. Notationally, $N(x) = \left\{y \in G \middle| yx^{-1} \in S\right\}$. Observe that from definition $N(e) = S$. For $x \in G$, define the mapping $\theta_x : G \to G$ by $\theta_x(y) = yx$, then θ_x is an automorphism of $\mathbb{T}_d$. We will call θ_x the *translation by* x. For a vertex $x \in \mathbb{T}_d$ and $x \neq e$, we denote by $|x|$, the length of the unique path from the root e to x and $|e| = 0$. Further, if $x \in \mathbb{T}_d$ and $x \neq e$, then we define $\overleftarrow{x}$ as the *parent* of x, that is, the penultimate vertex on the unique path from e to x.

Random Environment: Let $\mathcal{S} := \mathcal{S}_e$ be a collection of probability measures on the d elements of $N(e) = S$. To simplify the presentation and avoid various measurability issues, we assume that $\mathcal{S}$ is a Polish space (including the possibilities that $\mathcal{S}$ is finite or countably infinite). For each $x \in \mathbb{T}_d$, $\mathcal{S}_x$ is the push-forward of the space $\mathcal{S}$ under the translation θ_x, that is, $\mathcal{S}_x := \mathcal{S} \circ \theta_x^{-1}$. Note that the probabilities on $\mathcal{S}_x$ have support on $N(x)$. That is to say, an element $\omega(x, \cdot)$ of $\mathcal{S}_x$, is a probability measure satisfying

$$\omega(x, y) \geq 0 \quad \forall y \in \mathbb{T}_d \quad \text{and} \quad \sum_{y \in N(x)} \omega(x, y) = 1.$$

Let $\mathcal{B}_{\mathcal{S}_x}$ denote the Borel σ-algebra on $\mathcal{S}_x$. The *environment space* is defined as the measurable space $(\Omega, \mathcal{F})$ where

$$\Omega := \prod_{x \in \mathbb{T}_d} \mathcal{S}_x, \qquad \mathcal{F} := \bigotimes_{x \in \mathbb{T}_d} \mathcal{B}_{\mathcal{S}_x}. \tag{1}$$

An element $\omega \in \Omega$ will be written as $\left\{\omega(x, \cdot) \middle| x \in \mathbb{T}^d\right\}$. An environment distribution is a probability P on $(\Omega, \mathcal{F})$. We will denote by E the expectation taken with respect to the probability measure P.

Random Walk: Given an environment $\omega \in \Omega$, a random walk $(X_n)_{n \geq 0}$ is a time homogeneous Markov chain taking values in $\mathbb{T}_d$ with transition probabilities

$$\mathbf{P}_\omega\left(X_{n+1} = y \middle| X_n = x\right) = \omega(x, y).$$

Let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For each $\omega \in \Omega$, we denote by $\mathbf{P}_\omega^x$ the law induced by $(X_n)_{n \geq 0}$ on $\left((\mathbb{T}_d)^{\mathbb{N}_0}, \mathcal{G}\right)$, where $\mathcal{G}$ is the σ-algebra generated by the cylinder sets, such that

$$\mathbf{P}_\omega^x(X_0 = x) = 1. \tag{2}$$

The probability measure $\mathbf{P}_\omega^x$ is called the *quenched law* of the random walk $(X_n)_{n \geq 0}$, starting at x. We will use the notation $\mathbf{E}_\omega^x$ for the expectation under the quenched measure $\mathbf{P}_\omega^x$.

Following Zeitouni (2004), we note that for every $B \in \mathcal{G}$, the function

$$\omega \mapsto \mathbf{P}_\omega^x(B)$$

is $\mathcal{F}$-measurable. Hence, we may define the measure $\mathbb{P}^x$ on $\left(\Omega \times (\mathbb{T}_d)^{\mathbb{N}_0}, \mathcal{F} \otimes \mathcal{G}\right)$ by the relation

$$\mathbb{P}^x(A \times B) = \int_A \mathbf{P}_\omega^x(B)\, P(d\omega), \quad \forall A \in \mathcal{F}, B \in \mathcal{G}.$$

With a slight abuse of notation, we also denote the marginal of $\mathbb{P}^x$ on $(\mathbb{T}_d)^{\mathbb{N}_0}$ by $\mathbb{P}^x$, whenever no confusion occurs. This probability distribution is called the *annealed law* of the random walk $(X_n)_{n \geq 0}$, starting at x. We will use the notation $\mathbb{E}^x$ for the expectation under the annealed measure $\mathbb{P}^x$.

1.2. Main results

Throughout this paper we will assume that the following holds:

(A1) P is a product measure on $(\Omega, \mathcal{F})$ with "*identical*" marginals, that is, under P the random probability laws $\left\{\omega(x, \cdot) \middle| x \in \mathbb{T}^d\right\}$ are independent and "identically" distributed in the sense that

$$P \circ \theta_x^{-1} = P, \tag{3}$$

for all $x \in G$.

(A2) For all $1 \leq i \leq d$,

$$E\left[\left|\log \omega(e, s_i)\right|\right] < \infty. \tag{4}$$

It is worth noting that under this assumption $\omega(x, y) > 0$ almost surely (a.s.) with respect to the measure P for all $x \in \mathbb{T}_d$ and $y \in N(x)$.