STAPRO: 6967 Model 3Gsc pp. 1–4 (col. fig: NIL)

ARTICLE IN PRESS

Statistics and Probability Letters xx (xxxx) xxx–xxx

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Double extreme on joint sets for Gaussian random fields

Dan Cheng

North Carolina State University, United States

ARTICLE INFO

Article history:
Received 9 April 2014
Received in revised form 4 May 2014
Accepted 5 May 2014
Available online xxxx

Keywords:
Gaussian random fields
Excursion probability
Excursion set
Euler characteristic

ABSTRACT

For a centered Gaussian random field $X = \{X(t), t \in \mathbb{R}^N\}$, let T_1 and T_2 be two compact sets in $\mathbb{R}^N$ such that $I = T_1 \cap T_2 \neq \emptyset$ and denote by $\chi(A_u(I))$ the Euler characteristic of the excursion set $A_u(I) = \{t \in I : X(t) \geq u\}$. We show that under certain smoothness and regularity conditions, as $u \to \infty$, the joint excursion probability $\mathbb{P}\{\sup_{t\in T_1} X(t) \geq u, \sup_{s\in T_2} X(s) \geq u\}$ can be approximated by the expected Euler characteristic $\mathbb{E}\{\chi(A_u(I))\}$ such that the error is super-exponentially small.

1. Introduction

Throughout this paper, let $X = \{X(t), t \in \mathbb{R}^N\}$ be a real-valued centered Gaussian random field. Let T_1 and T_2 be two compact sets in $\mathbb{R}^N$ and denote $I = T_1 \cap T_2$ and $T = T_1 \cup T_2$. The intersection I is always assumed nonempty, unless stated otherwise. For a set $D \subset \mathbb{R}^N$, denote by $\chi(D)$ the Euler characteristic of D and by $A_u(D) = \{t \in D : X(t) \geq u\}$ the excursion set above level $u \in \mathbb{R}$ (cf. Adler and Taylor, 2007).

The excursion probability $\mathbb{P}\{\sup_{t\in T} X(t) \geq u\}$ has been extensively studied due to the significant value in both theory and applications in many areas. We refer to monographs by Piterbarg (1996), Adler and Taylor (2007) and Azaïs and Wschebor (2009) for both the history and recent developments on studying the excursion probability. As a natural extension, the joint excursion probability $\mathbb{P}\{\sup_{t\in T_1} X(t) \geq u, \sup_{s\in T_2} X(s) \geq u\}$ was also investigated recently, see Ladneva and Piterbarg (2000), Piterbarg and Stamatovic (2005), Anshin (2006) and Debicki et al. (2010). In particular, motivated by the expected Euler characteristic approximation (Adler and Taylor, 2007), and Cheng (2013) showed that for a unit-variance Gaussian field X, assuming $I = \emptyset$ and certain smoothness and regularity conditions, there exists $\alpha > 0$ such that as $u \to \infty$, the following approximation holds:

$$\mathbb{P}\left\{\sup_{t\in T_1} X(t) \geq u, \sup_{s\in T_2} X(s) \geq u\right\} = \mathbb{E}\{\chi(A_u(T_1) \times A_u(T_2))\} + o\left(\exp\left\{-\alpha u^2 - \frac{u^2}{1+\rho(T_1,T_2)}\right\}\right), \qquad (1.1)$$

where $\rho(T_1, T_2) = \sup_{t\in T_1, s\in T_2} \mathbb{E}\{X(t)X(s)\}$.

Notice that, when applying the double-sum method, in order to avoid degeneracy, the existing results on joint excursion probability, including (1.1), require T_1 and T_2 to be disjoint. Here, we consider the alternative case where T_1 and T_2 intersect, i.e. $I \neq \emptyset$. Our main result is that, for a certain class of smooth Gaussian fields, there exists $\alpha > 0$ such that as $u \to \infty$, the

E-mail address: cheng@stt.msu.edu.

http://dx.doi.org/10.1016/j.spl.2014.05.001

Please cite this article in press as: Cheng, D., Double extreme on joint sets for Gaussian random fields. Statistics and Probability Letters (2014), http://dx.doi.org/10.1016/j.spl.2014.05.001

following approximation holds:

$$\mathbb{P}\left\{\sup_{t\in T_1} X(t) \geq u, \sup_{s\in T_2} X(s) \geq u\right\} = \mathbb{E}\{\chi(A_u(I))\} + o\left(\exp\left\{-\alpha u^2 - \frac{u^2}{2\sigma_I^2}\right\}\right), \tag{1.2}$$

where $\sigma_I^2 = \sup_{t\in I} \mathrm{Var}(X(t))$. Note that the expectations in (1.1) and (1.2) can be computed explicitly via the Kac–Rice formula (cf. Adler and Taylor, 2007).

2. Smooth Gaussian random fields with constant variance

We first consider the case where X has constant variance, implying $\mathrm{Var}(X(t)) \equiv \sigma_I^2$. When $X(\cdot) \in C^2(\mathbb{R}^N)$ almost surely, we write $\frac{\partial X(t)}{\partial t_i} = X_i(t)$ and $\frac{\partial^2 X(t)}{\partial t_i \partial t_j} = X_{ij}(t)$ and denote by $\nabla X(t)$ the vector $(X_1(t), \ldots, X_N(t))^T$. To make the field X a Morse function almost surely, which is required for showing the expected Euler characteristic approximation (cf. Adler and Taylor, 2007), we need the following smoothness and regularity conditions.

(**C**1). $X(\cdot) \in C^2(T)$ almost surely and there exist constants $L, \eta, \delta > 0$ such that

$$\mathbb{E}(X_{ij}(t) - X_{ij}(s))^2 \leq L|\log\|t-s\||^{-(1+\eta)}, \quad \forall \|t-s\| \leq \delta,\ i,j = 1,\ldots,N.$$

(**C**2). For every $t \in T$, $(X(t), \nabla X(t), X_{ij}(t), 1 \leq i \leq j \leq N)$ is non-degenerate.

Before stating the result, let us recall the term "*locally convex*" defined in Adler and Taylor (2007), p. 189. The rigorous definition is omitted here, however, it should be mentioned that a convex set is locally convex and locally convexity is a generalization of convexity.

Theorem 2.1. *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a constant-variance Gaussian random field satisfying* (**C**1) *and* (**C**2). *If all the sets T_1, T_2 and T are locally convex and piecewise smooth, then there exists $\alpha > 0$ such that* (1.2) *holds as $u \to \infty$.*

Proof. Since $T = T_1 \cup T_2$, we can write

$$\mathbb{P}\left\{\sup_{t\in T_1} X(t) \geq u, \sup_{s\in T_2} X(s) \geq u\right\} = \mathbb{P}\left\{\sup_{t\in T_1} X(t) \geq u\right\} + \mathbb{P}\left\{\sup_{t\in T_2} X(t) \geq u\right\} - \mathbb{P}\left\{\sup_{t\in T} X(t) \geq u\right\}. \tag{2.1}$$

Applying the expected Euler characteristic approximation (cf. Adler and Taylor, 2007, Theorem 14.3.3) to all three terms in the second line of (2.1), we obtain that there exists $\alpha > 0$ such that

$$\begin{aligned}
&\mathbb{P}\left\{\sup_{t\in T_1} X(t) \geq u, \sup_{s\in T_2} X(s) \geq u\right\} \\
&\quad = \mathbb{E}\{\chi(A_u(T_1))\} + \mathbb{E}\{\chi(A_u(T_2))\} - \mathbb{E}\{\chi(A_u(T))\} + o\left(\exp\left\{-\alpha u^2 - \frac{u^2}{2\sigma_I^2}\right\}\right) \\
&\quad = \mathbb{E}\{\chi(A_u(T_1)) + \chi(A_u(T_2)) - \chi(A_u(T))\} + o\left(\exp\left\{-\alpha u^2 - \frac{u^2}{2\sigma_I^2}\right\}\right).
\end{aligned} \tag{2.2}$$

Note that $A_u(T) = A_u(T_1 \cup T_2) = A_u(T_1) \cup A_u(T_2)$. Therefore, by the additivity of Euler characteristic (i.e., $\chi(D_1 \cup D_2) = \chi(D_1) + \chi(D_2) - \chi(D_1 \cap D_2)$ for any two suitable sets D_1 and D_2, see for example Adler and Taylor (2007), Chapter 6)

$$\chi(A_u(T_1)) + \chi(A_u(T_2)) - \chi(A_u(T)) = \chi(A_u(T_1) \cap A_u(T_2)) = \chi(A_u(I)). \tag{2.3}$$

Plugging (2.3) into (2.2) gives the desired result. $\square$

The result in Theorem 2.1 can be intuitively seen by noting that for any distinct points $t, s \in T$, the joint probability $\mathbb{P}\{X(t) \geq u, X(s) \geq u\}$ is negligible or super-exponentially small when compared with $\mathbb{E}\{\chi(A_u(I))\}$ as $u \to \infty$. Thus $\mathbb{P}\{\sup_{t\in I} X(t) \geq u\}$ makes the major contribution to the joint excursion probability $\mathbb{P}\{\sup_{t\in T_1} X(t) \geq u, \sup_{s\in T_2} X(s) \geq u\}$, and meanwhile, it can be approximated by the expected Euler characteristic $\mathbb{E}\{\chi(A_u(I))\}$.

Here, it is also valuable to compare the result in Theorem 2.1 with Pickands' approximation, which is another common approximation for excursion probability (see for example Piterbarg, 1996). For simplicity, assume that X is stationary with covariance $r(t)$ satisfying $r(t) = 1 - \|t\|^\beta$ as $\|t\| \to 0$, where $\beta \in (0, 2]$ is some constant. Then, for any N-dimensional Jordan measurable set $D \subset \mathbb{R}^N$, there exists some function g independent of D such that $\mathbb{P}\{\sup_{t\in D} X(t) \geq u\} = \mathrm{Vol}(D)g(u)(1 + o(1))$ as $u \to \infty$. Applying this approximation to (2.1), we can approximate the joint excursion probability by $[\mathrm{Vol}(T_1) + \mathrm{Vol}(T_2) - \mathrm{Vol}(T)]g(u)$. But it becomes meaningless if $\mathrm{Vol}(T_1) + \mathrm{Vol}(T_2) - \mathrm{Vol}(T) = 0$ or the dimension of I is less than N. In contrast, this problem does not exists in Theorem 2.1 since the expected Euler characteristic approximation takes into account the geometry of all lower dimensional faces. This phenomenon may also promote us to find more accurate approximations for the excursion probability of non-smooth Gaussian fields.

Please cite this article in press as: Cheng, D., Double extreme on joint sets for Gaussian random fields. Statistics and Probability Letters (2014), http://dx.doi.org/10.1016/j.spl.2014.05.001