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Inference for 2-D GARCH models

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ABSTRACT

The purpose of this paper is, in the first step, to consider a class of GMM estimators with interesting asymptotic properties and a reasonable number of computations for two dimensionally indexed Generalized Autoregressive Conditional Heteroscedasticity (GARCH) model. In the second step, we use the central limit theorem of Huang (1992) for spatial martingale differences to establish the LAN property for general two-dimensional discrete models on a regular grid with Gaussian errors. We then apply this result to the spatial GARCH model and derive the limit distribution of the maximum likelihood estimators of the parameters. Results of numerical simulations are presented.

1. Introduction

The introduction of the Autoregressive Conditional Heteroscedasticity (ARCH) model in the famous paper of Engle (1982) was a natural starting point in modeling the temporal dependencies in the conditional variance of financial time series. This model allows the variance to depend on the past of the random process. Numerous variants and extensions of this model have been proposed. Generalized ARCH (GARCH) model is the main natural extension of this model, the passage has been done in a way that is similar to the passage from the AR model to the ARMA one. A large part of the financial literature is devoted to one-dimensional GARCH model; see for example Bollerslev (1986), Bollerslev et al. (1994), Palm (1996), Shephard (1996). Next, this GARCH model has seen many extensions with the introduction of lagged values of the variance or models allowing to take into account the phenomena of asymmetry such as EGARCH models (Exponential GARCH) proposed by Nelson (1991), TGARCH models (Threshold GARCH) proposed by Zakoïan (1994), or again DCC-MVGARCH models (Multivariate GARCH with Dynamically Conditional Correlation) proposed by Engle and Sheppard (2001). The treatment of spatial interaction (dependence) and spatial structure (heterogeneity) in practice may be modeled by some random fields $(X_{\mathbf{t}})_{\mathbf{t}\in\mathbb{Z}^d}$. Noiboar and Cohen (2005) had the idea of extending the one-dimensional GARCH model into two-dimensions in order to take into account the variability of the variance through the space. They could also show that the two-dimensional GARCH model generalizes the causal Gauss Markov Random Field (GMRF), largely used in cluster modeling with the disadvantage of having a constant conditional variance through the space which makes the use of a GARCH cluster modeling better than the use of a GMRF one.

This phenomena is often found on natural images because they are corrupted due to several factors, such as performance of imaging sensors and characteristics of the transmission channel (Amirmazlaghani and Amindavar, 2010). On the other hand, research on statistical properties of images' wavelet coefficients have shown that the marginal distribution of wavelet Q2 coefficients are highly kurtotic, and can be described using suitable heavy-tailed distribution (cf. Achim et al., 2003). Indeed, Amirmazlaghani et al. (2009) shown that the subband decomposition of SAR images has significantly non-Gaussian statistics Q3 that are best described by the 2-D GARCH model.

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It should be noted that statistical and probabilistic properties as well as building the parameter estimators have gained more attention for the spatial linear models than the nonlinear one, despite of the well known nonlinearity structure of many spatial series, this is partly due to the fact that the existence of spatial dependence creates difficulties for building such estimators. In spite of some observed similarities, the temporal and spatial situations do not lead to the same consequences in the development of regression models in general (cf. Kharfouchi, 2012).

Local Asymptotic Normality (LAN) and Generalized Method of Moment (GMM) provide theoretical frameworks which have become standard and convenient tools for proving efficiency of estimators. So, the purpose of this paper is, in the first step, to consider a class of GMM estimators with attractive asymptotic properties and a reasonable computational burden. In the second step, we use the central limit theorem of Huang (1992) for spatial martingale differences to establish the LAN property for general two-dimensional discrete models on a regular grid with Gaussian errors. We then apply this result to the spatial GARCH model and derive the limit distribution of the maximum likelihood estimators of the parameters. A major advantage of each of the above estimators is that they are explicit and do not require a numerical optimization algorithm. It is noted that often, the spatial sites on which data are collected are irregularly positioned, but with the increasing use of computer technology, measured data on a regular grid on a continuous scale are becoming more and more common. As indicated by Tjostheim (1978), in some situations at least, irregularly spaced data may be replaced by data on a regular grid using interpolation techniques (cf. Delfiner and Delhomme, 1975). Hence in this work, we are interested in spatial GARCH processes on an infinite regular lattice, they are unilateral by construction, hence they are quite analogous to the well understood GARCH time series models defined on the integers.

Throughout, $I_{(k)}$ denotes the identity matrix of order k, $O_{(k)}$ denotes the $k \times k$ matrix whose entries are zeros. $Vec(M)$ denotes the vector obtained by setting down the column of M underneath each other. Also, recall that $M^{\otimes 2} = M \otimes M$ denotes the usual Kronecker product of M with itself. All vectors are underlined except those of $\mathbb{Z}^2$ are written in bold, so $\mathbf{0} = (0, 0)$ $\mathbf{1} = (1, 1)$, $\mathbf{e}_0 = (0, 1)$, $\mathbf{e}_1 = (1, 0)$ and $\mathbf{n} = (n_1, n_2)$. For any $\mathbf{s} = (s_1, s_2)$, $\mathbf{t} = (t_1, t_2) \in \mathbb{Z}^2$, we write $\mathbf{s} \preccurlyeq \mathbf{t}$ if and only if $[(s_1 < t_1) \vee (s_1 = t_1 \text{ and } s_2 < t_2)]$. However for $\mathbf{a} = (a_1, a_2)$, $\mathbf{b} = (b_1, b_2) \in \mathbb{Z}^2$ such that $\mathbf{a} \preceq \mathbf{b}$, the following indexing subset in $\mathbb{Z}^2$ will be considered $S[\mathbf{a}, \mathbf{b}] := \left\{(l, m) \in \mathbb{Z}^2,\ (a_1, a_2) \preceq (l, m) \preceq (b_1, b_2)\right\}$, $S\,]\mathbf{a}, \mathbf{b}] = S[\mathbf{a}, \mathbf{b}] \setminus \{\mathbf{a}\}$ and $S[\mathbf{a}, \mathbf{b}[= S[\mathbf{a}, \mathbf{b}] \setminus \{\mathbf{b}\}$.

The paper is organized as follows. The spatial GARCH model is presented and a sufficient condition for $2mth$-order stationarity is given in Section 2. Section 3 proposes the GMM methodology to estimate the parameters of the 2-D GARCH model. The strong consistency and the asymptotic normality of the estimators are derived under optimal conditions. In Section 4, we use the central limit theorem of Huang (1992) for spatial martingale differences to establish the LAN property for a general two-dimensional discrete model. We then apply this result to the 2-D GARCH model. Finally, results of numerical simulations are presented in Section 5.

2. Spatial GARCH models

The 2-D GARCH (p_1, p_2, q_1, q_2) process is defined as:

$$\begin{cases} \varepsilon_{\mathbf{t}} = \sqrt{h_{\mathbf{t}}}\eta_{\mathbf{t}} \\ h_{\mathbf{t}} = \alpha_0 + \sum\limits_{\mathbf{k} \in S]\mathbf{0},(p_1,p_2)]} \alpha_{\mathbf{k}} \varepsilon^2_{\mathbf{t}-\mathbf{k}} + \sum\limits_{\mathbf{l} \in S]\mathbf{0},(q_1,q_2)]} \beta_{\mathbf{l}} h_{\mathbf{t}-\mathbf{l}}, \end{cases} \tag{2.1}$$

$\alpha_0 > 0$, $\alpha_{\mathbf{k}} \geq 0$, $\mathbf{k} \in S\,]\mathbf{0}, (p_1, p_2)]$ and $\beta_{\mathbf{l}} \geq 0$, $\mathbf{l} \in S\,]\mathbf{0}, (q_1, q_2)]$ where $h_{\mathbf{t}}$ is the conditional variance of $\varepsilon_{\mathbf{t}}$ (i.e., $\varepsilon_{\mathbf{t}} \setminus \psi_{\mathbf{t}} \sim \mathcal{N}(0, h_{\mathbf{t}})$) and $\psi_{\mathbf{t}}$ denotes all the information, namely $\psi_{\mathbf{t}} = \left\{(\varepsilon_{\mathbf{t}-\mathbf{k}})_{\mathbf{k} \preceq (p_1,p_2)}, (h_{\mathbf{t}-\mathbf{l}})_{\mathbf{l} \preceq (q_1,q_2)}\right\}$, and $\eta_{\mathbf{t}} \sim \mathcal{N}(0, 1)$ is a stochastic 2-D process independent of $h_{\mathbf{k}}$, $\forall \mathbf{k} \preceq \mathbf{t}$.

Similar to the time series case, there is a necessary and sufficient condition for the model (2.1) to be weakly stationary that is,

$$\sum_{\mathbf{k} \in S]\mathbf{0},(p_1,p_2)]} \alpha_{\mathbf{k}} + \sum_{\mathbf{l} \in S]\mathbf{0},(q_1,q_2)]} \beta_{\mathbf{l}} < 1. \tag{2.1$'$}$$

The estimation of the parameters of a model like (2.1) has been studied by Noiboar and Cohen (2005) but not from an asymptotic point of view, they proposed the method of maximum likelihood.

As we need *fourth*-order stationarity of the GARCH model for the rest of the work, in the following theorem we obtain a sufficient condition for $2mth$-order stationarity. First, let $\underline{z}_{\mathbf{t}} = \left(\left(\underline{\varepsilon}^2_{\mathbf{t}-\mathbf{k}}\right)_{\mathbf{k} \in S[\mathbf{0},(p_1,p_2)]}, \left(\underline{h}_{\mathbf{t}-\mathbf{l}}\right)_{\mathbf{l} \in S[\mathbf{0},(q_1,q_2)]}\right)' \in \mathbb{R}^{P+Q}$ ($P = (p_1 + 1)(p_2 + 1)$ and $Q = (q_1 + 1)(q_2 + 1)$) and $\underline{b}_{\mathbf{t}} = \left(\alpha_0 \eta^2_{\mathbf{t}}, O_{1\times(P-1)}, \alpha_0, O_{1\times(Q-1)}\right)' \in \mathbb{R}^{P+Q}$. Then (2.1) is equivalently written as a vector stochastic equation

$$\underline{z}_{\mathbf{t}} = \underline{b}_{\mathbf{t}} + A_1(\mathbf{t})\,\underline{z}_{\mathbf{t}-\mathbf{e}_0} + A_2(\mathbf{t})\,\underline{z}_{\mathbf{t}-\mathbf{e}_1}, \tag{2.2}$$

where $A_1(\mathbf{t})$ and $A_2(\mathbf{t})$ are $(P + Q) \times (P + Q)$ random matrices whose components are depending on $(\alpha_{\mathbf{k}})_{\mathbf{k} \in S]\mathbf{0},(p_1,p_2)]}$, $(\beta_{\mathbf{l}})_{S]\mathbf{0},(q_1,q_2)]}$ and $\eta^2_{\mathbf{t}}$.