Statistics and Probability Letters xx (xxxx) xxx–xxx

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Rate of Poisson approximation for nearest neighbor counts in large-dimensional Poisson point processes

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ARTICLE INFO

Article history:
Received 29 October 2013
Received in revised form 22 May 2014
Accepted 27 May 2014
Available online xxxx

MSC:
primary 60D05
secondary 60G55

Keywords:
Typical point
Typical cell
Voronoi tessellation
Total variation distance

ABSTRACT

Consider two independent homogeneous Poisson point processes Π of intensity λ and Π' of intensity λ' in d-dimensional Euclidean space. Let $q_{k,d}$, $k = 0, 1, \ldots$, be the fraction of Π-points which are the nearest Π-neighbor of precisely k Π'-points. It is known that as $d \to \infty$, the $q_{k,d}$ converge to the Poisson probabilities $e^{-\lambda'/\lambda}(\lambda'/\lambda)^k/k!$, $k = 0, 1, \ldots$. We derive the (sharp) rate of convergence $d^{-1/2}(4/3\sqrt{3})^d$, which is related to the asymptotic behavior of the variance of the volume of the typical cell of the Poisson–Voronoi tessellation generated by Π. An extension to the case involving more than two independent Poisson point processes is also considered.

In this note, we consider two independent homogeneous Poisson point processes Π of intensity λ and Π' of intensity λ' in d-dimensional Euclidean space $\mathbb{R}^d$. Let $p_{k,d}$, $k = 0, 1, \ldots$, be the fraction of Π-points which are the nearest Π-neighbor of precisely k other Π-points, and $q_{k,d}$, $k = 0, 1, \ldots$, the fraction of Π-points which are the nearest Π-neighbor of precisely k Π'-points. Here for given $\mathbf{v} \in \mathbb{R}^d$, a point $Q \in \Pi$ is called the nearest Π-neighbor of $\mathbf{v}$ if $\|Q - \mathbf{v}\|_d < \|\mathbf{u} - \mathbf{v}\|_d$ for all $\mathbf{u} \in \Pi \setminus \{\mathbf{Q}\}$ where $\|\cdot\|_d$ denotes the Euclidean norm in $\mathbb{R}^d$. By ergodic-type arguments, $p_{k,d}$ and $q_{k,d}$ are well defined. In their Theorems 5 and 10, Newman et al. (1983) proved that

$$\lim_{d\to\infty} p_{k,d} = e^{-1}/k! \quad \text{and} \quad \lim_{d\to\infty} q_{k,d} = e^{-\rho}\rho^k/k!, \tag{1}$$

where $\rho = \lambda'/\lambda$. (See also Newman and Rinott, 1985.)

The limit results in (1) can also be formulated in terms of a "typical point" Q of Π, to be translated so that $Q = \mathbf{0} = (0, \ldots, 0)$, the origin in $\mathbb{R}^d$. We will refer to $\mathbf{0}$ as the typical point of Π. (Note that since Π is a Poisson point process, its Palm distribution at $\mathbf{0}$ is equivalent to the distribution of Π with an independently added point at $\mathbf{0}$; see e.g. Daley and Vere-Jones (2007, Proposition 13.1.VII).) Let M_d be the number of Π-points which have $\mathbf{0}$ as their nearest Π-neighbor, and $N_{\rho,d}$ the number of Π'-points which have $\mathbf{0}$ as their nearest Π-neighbor, i.e.

$$\begin{aligned} M_d &= \#\{\mathbf{u} \in \Pi : \|\mathbf{u} - \mathbf{0}\|_d < \|\mathbf{u} - \mathbf{v}\|_d \text{ for all } \mathbf{v} \in \Pi \setminus \{\mathbf{u}\}\}, \\ N_{\rho,d} &= \#\{\mathbf{u}' \in \Pi' : \|\mathbf{u}' - \mathbf{0}\|_d < \|\mathbf{u}' - \mathbf{v}\|_d \text{ for all } \mathbf{v} \in \Pi\} \\ &= \#(\Pi' \cap C_d), \end{aligned} \tag{2}$$

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http://dx.doi.org/10.1016/j.spl.2014.05.014
0167-7152/© 2014 Published by Elsevier B.V.

Please cite this article in press as: Yao, Y.-C., Rate of Poisson approximation for nearest neighbor counts in large-dimensional Poisson point processes. Statistics and Probability Letters (2014), http://dx.doi.org/10.1016/j.spl.2014.05.014

where

$$C_d = \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x} - \mathbf{0}\|_d < \|\mathbf{x} - \mathbf{u}\|_d \text{ for all } \mathbf{u} \in \Pi\}, \tag{3}$$

the (typical) Voronoi cell centered at $\mathbf{0}$ generated by $\Pi \cup \{\mathbf{0}\}$ (cf. Okabe et al., 2000). Note that $\mathcal{L}(M_d)$, the distribution of M_d, is independent of λ and λ', and $\mathcal{L}(N_{\rho,d})$ depends on λ and λ' only through $\rho = \lambda'/\lambda$. Then (1) is equivalent to the limit results that as $d \to \infty$, M_d and $N_{\rho,d}$ converge in distribution to $Po(1)$ and $Po(\rho)$, respectively, where $Po(\rho)$ denotes the Poisson distribution with mean ρ.

In the present note, we derive the rate of convergence for $N_{\rho,d}$ as stated below.

Theorem 1. *For any given $\rho_0 > 0$, there exists a constant $c_1(\rho_0) > 0$ such that for all $1 \le d < \infty$ and $0 < \rho \le \rho_0$,*

$$c_1(\rho_0)\rho^2 \frac{1}{\sqrt{d}} \left(\frac{4}{3\sqrt{3}}\right)^d \le d_{TV}(\mathcal{L}(N_{\rho,d}), Po(\rho)) \le c_2\rho(1 - e^{-\rho}) \frac{1}{\sqrt{d}} \left(\frac{4}{3\sqrt{3}}\right)^d, \tag{4}$$

where d_{TV} denotes the total variation distance and $c_2 > 0$ is a constant (independent of ρ_0).

Proof. Without loss of generality, assume $\lambda = 1$, so that $\rho = \lambda'/\lambda = \lambda'$. Note that $N_{\rho,d}$ has a mixed Poisson distribution. By (2) and (3), the conditional distribution of $N_{\rho,d}$ given $\mu_d(C_d) = v$ is $Po(\lambda' v) = Po(\rho v)$ where $\mu_d(S)$ denotes the d-dimensional Lebesgue measure (volume) of measurable $S \subset \mathbb{R}^d$. Also, $\mathrm{E}[\mu_d(C_d)] = 1/\lambda = 1$, and by Alishahi and Sharifitabar (2008, Theorem 3.1),

$$\operatorname{var}(\mu_d(C_d)) \le \frac{c_2}{\lambda^2} \frac{1}{\sqrt{d}} \left(\frac{4}{3\sqrt{3}}\right)^d = c_2 \frac{1}{\sqrt{d}} \left(\frac{4}{3\sqrt{3}}\right)^d \quad \text{for all } 1 \le d < \infty,$$

for some constant $c_2 > 0$. By Barbour et al. (1992, Theorem 1.C(ii)), we have

$$\begin{aligned} d_{TV}(\mathcal{L}(N_{\rho,d}), Po(\rho)) &\le \rho^{-1}(1 - e^{-\rho})\operatorname{var}(\rho\mu_d(C_d)) \\ &\le c_2\rho(1 - e^{-\rho}) \frac{1}{\sqrt{d}} \left(\frac{4}{3\sqrt{3}}\right)^d, \end{aligned}$$

establishing the upper bound.

To derive the lower bound, fix a (large) $u > 0$ and consider the ball of volume u centered at $\mathbf{0}$, denoted by $B = B_{u,d}$. Applying Lemma 1 below with $\alpha_1 = \rho\mu_d(C_d \cap B) \le \rho u$, $\alpha_2 = \rho\mu_d(C_d \setminus B)$, $\beta_1 = \mathrm{E}[\alpha_1] = \rho \mathrm{E}[\mu_d(C_d \cap B)] \le \rho u$, $\beta_2 = \mathrm{E}[\alpha_2] = \rho\mathrm{E}[\mu_d(C_d \setminus B)] = \rho - \beta_1$, we have

$$\begin{aligned} d_{TV}(\mathcal{L}(N_{\rho,d}), Po(\rho)) &\ge P(N_{\rho,d} = 0) - e^{-\rho} \\ &= \mathrm{E}[e^{-\rho\mu_d(C_d)}] - e^{-\rho} \\ &= \mathrm{E}\{e^{-\alpha_1-\alpha_2} - e^{-\beta_1-\beta_2}\} \\ &\ge \mathrm{E}\{e^{-\beta_1-\beta_2}[h_1(\rho u)(\alpha_1 - \beta_1)^2 - (\alpha_1 - \beta_1)] - e^{-\alpha_1-\beta_2}(\alpha_2 - \beta_2)\} \\ &= e^{-\rho}h_1(\rho u)\operatorname{var}(\alpha_1) - e^{-\beta_2}\mathrm{E}[e^{-\alpha_1}(\alpha_2 - \beta_2)] \\ &\ge e^{-\rho}h_1(\rho u)\operatorname{var}(\rho\mu_d(C_d \cap B)) \\ &\ge e^{-\rho}h_1(\rho u)\rho^2 c \frac{1}{\sqrt{d}} \left(\frac{4}{3\sqrt{3}}\right)^d, \end{aligned}$$

where the last two inequalities follow from Lemma 2 (with $S_1 = B$ and $S_2 = \mathbb{R}^d \setminus B$) and Lemma 3 (with $u \ge u_0$ and $c > 0$ and $u_0 < \infty$ appearing in the statement of Lemma 3). Noting that h_1 is nonincreasing, we have for all $1 \le d < \infty$ and $0 < \rho \le \rho_0$,

$$\begin{aligned} d_{TV}(\mathcal{L}(N_{\rho,d}), Po(\rho)) &\ge ce^{-\rho_0}h_1(\rho_0 u)\rho^2 \frac{1}{\sqrt{d}} \left(\frac{4}{3\sqrt{3}}\right)^d \\ &= c_1(\rho_0)\rho^2 \frac{1}{\sqrt{d}} \left(\frac{4}{3\sqrt{3}}\right)^d, \end{aligned}$$

where $c_1(\rho_0) = ce^{-\rho_0}h_1(\rho_0 u)$. The proof is complete. □

Remark 1. The lower and upper bounds in (4) may be expressed as

$$\inf_{1 \le d < \infty, 0 < \rho \le \rho_0} d_{TV}(\mathcal{L}(N_{\rho,d}), Po(\rho)) \Bigg/ \left[\rho^2 \frac{1}{\sqrt{d}} \left(\frac{4}{3\sqrt{3}}\right)^d\right] > 0,$$

$$\sup_{1 \le d < \infty, \rho > 0} d_{TV}(\mathcal{L}(N_{\rho,d}), Po(\rho)) \Bigg/ \left[\rho(1 - e^{-\rho}) \frac{1}{\sqrt{d}} \left(\frac{4}{3\sqrt{3}}\right)^d\right] < \infty.$$

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