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First-passage times of regime switching models

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ABSTRACT

The probability of a stochastic process to first breach an upper and/or a lower level is an important quantity for optimal control and risk management. We present those probabilities for regime switching Brownian motion. In the 2- and 3-state model, the Laplace transform of the (single and double barrier) first-passage times is – up to the roots of a polynomial of degree 4 (respectively 6) – derived in closed-form by solving the matrix Wiener–Hopf factorization.[1] This extends single barrier results in the 2-state model by Guo (2001b). If the quotient of drift and variance is constant over all states, we show that the Laplace transform can even be inverted analytically.

After Hamilton (1989)'s seminal work, the natural and intuitive idea of regime changes has found applications in a variety of fields, ranging from biology, physics, finance, and insurance to disciplines like hydrology. Conceptionally, regime switching models are rather simple (conditional on the regimes, the innovations are normally distributed) and thus analytically tractable. Nevertheless, they can generate many non-linear effects like heavy tails or volatility clusters. Regime switching models allow us to depart from the unsatisfactory assumption of stationary increments in Lévy models and are thus a tractable tool to include long-term trends and structural breaks.

This paper derives analytical expressions for the single and double barrier first-passage time probabilities of regime switching models. Therefore, the matrix Wiener–Hopf factorization introduced by, for example, London et al. (1982), Kennedy and Williams (1990), Barlow et al. (1990), Rogers (1994), Asmussen (1995), and Jiang and Pistorius (2008) is solved analytically for the 2- and 3-state model. This yields closed-form results for the Laplace transform of the first-passage times. Imposing a parameter restriction (the quotient of drift and variance is constant over all states), we are even able to invert this Laplace transform analytically. This contributes to the analytical tractability of regime switching models and might help to increase the popularity of this model class. Related to this work is Guo (2001b) who derived the Laplace transform of the single barrier first-passage time in the 2-state model, and Jiang and Pistorius (2008) who present the single and double barrier first-passage time probabilities in terms of the matrix Wiener–Hopf factorization.

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[1] The matrix Wiener–Hopf factors of regime switching models are defined via a set of quadratic matrix equations (see, e.g., London et al., 1982; Barlow et al., 1990; Kennedy and Williams, 1990; Rogers and Shi, 1994; Asmussen, 1995). This concept was expanded to regime switching jump diffusions by Jiang and Pistorius (2008).

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Many authors recently worked on numerical techniques to derive the first-passage time probabilities of regime switching models. Boyle and Draviam (2007), Kim et al. (2008) (and many others) solve the first-passage time PDE numerically. Hieber and Scherer (2010) and Henriksen (2011) use a conditional Monte-Carlo technique called Brownian bridge algorithm. Furthermore, some authors work on (matrix) Wiener–Hopf factorizations (see, e.g., London et al., 1982, Barlow et al., 1990, Boyarchenko and Levendorskiĭ, 2008, Jiang and Pistorius, 2008, Kudryavtsev and Levendorskiĭ, 2012, Mijatović and Pistorius, 2013, Fourati, 2012, and many others) and solve them numerically (see, e.g., Rogers and Shi, 1994, Boyarchenko and Levendorskiĭ, 2008, Kudryavtsev, 2010). Kou and Wang (2003) point out that "in general, explicit calculation of the Wiener–Hopf factorization is difficult". Its derivation has turned out to be possible in the related case of exponential jump-diffusion models, i.e. the Cramér–Lundberg model, single, double, and hyper-exponential jump-diffusion models (see, e.g., Mordecki, 1999, Rogers, 2000, Kou and Wang, 2003, Avram et al., 2003 and Cai, 2009). Up to a time change, those models can be seen as special cases of regime switching models: positive or negative exponential jumps are included as an additional state with zero volatility and positive, respectively negative, drift (a technique called "*fluid embedding*", see, e.g., Jiang and Pistorius, 2008).

The paper is organized as follows: Section 1 introduces regime switching models; Section 2 the matrix Wiener–Hopf factorization. The main theoretical results on the first-passage times of regime switching models are given in Section 3. Section 4 presents a numerical example. Finally, Section 5 concludes.

1. Model description

On the filtered probability space $(\Omega, \mathbb{F}, \mathcal{F}, \mathbb{P})$, we consider the process $B = \{B_t\}_{t\geq 0}$ described by the stochastic differential equation (sde)

$$dB_t = \mu_{Z_t} dt + \sigma_{Z_t} dW_t, \qquad B_0 = x, \tag{1}$$

where $Z = \{Z_t\}_{t\geq 0} \in \{1, 2, \dots, M\}$ is a time-homogeneous Markov chain with intensity matrix[2] Q_0 and $W = \{W_t\}_{t\geq 0}$ an independent Brownian motion. The initial value is $B_0 = x \in \mathbb{R}$. The filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ is generated by the pair (W, Z), i.e. $\mathcal{F}_t = \sigma\{W_s, Z_s : 0 \leq s \leq t\}$. The time to a state change from the current state i is an exponential random variable with intensity parameter $Q_0(i, i)$. The probability of moving to state $j \neq i$ is $-Q_0(i, j)/Q_0(i, i)$. The model is fully determined if an initial state (or, more generally, an initial distribution $\pi_0 := \big(\mathbb{P}(Z_0 = 1), \mathbb{P}(Z_0 = 2), \dots, \mathbb{P}(Z_0 = M)\big)$ on the states) is defined. The characteristic function of a regime switching model is given by (see, e.g., Buffington and Elliott, 2002 and Elliott et al., 2005)

$$\phi_t(u) := \mathbb{E}\left[\exp\big(\boldsymbol{i}u(B_t - x)\big)\right] = \left\langle \exp\left(Q_0 t + \begin{pmatrix} \boldsymbol{i}u\mu_1 - \frac{1}{2}\sigma_1^2 u^2 & 0 & \cdots \\ 0 & \cdots & 0 \\ \cdots & 0 & \boldsymbol{i}u\mu_M - \frac{1}{2}\sigma_M^2 u^2 \end{pmatrix} t \right) \pi_0', \mathbf{1} \right\rangle, \tag{2}$$

where $\exp(\cdot)$ denotes the matrix exponential function, $'$ transpose, $\mathbf{1}$ a vector of ones of appropriate size, and $\langle \cdot, \cdot \rangle$ the scalar product. The first-passage times on two constant barriers $b < B_0 = x < a$ are defined as

$$T_{ab} := \begin{cases} \inf\{t \geq 0 : B_t \notin (b, a)\}, & \text{if such a } t \text{ exists,} \\ \infty, & \text{if } B_t \text{ never hits the barriers.} \end{cases} \tag{3}$$

Here, T_{ab} is the first time the Brownian motion B_t hits one of the two barriers a and b. Further denote

$$T_{ab}^{+} := T_{ab}, \quad \text{if } B_{T_{ab}} \geq a, \qquad T_{ab}^{-} := T_{ab}, \quad \text{if } B_{T_{ab}} \leq b. \tag{4}$$

The Laplace transform of the first-passage time is defined as

$$\Psi_{ab}^{\pm}(u) := \mathbb{E}\left[\exp\left(-uT_{ab}^{\pm}\right)\right]. \tag{5}$$

2. Review of the matrix Wiener–Hopf factorization

The rudiment of this work is the matrix Wiener–Hopf factorization as introduced by London et al. (1982), Kennedy and Williams (1990), Barlow et al. (1990), Rogers (1994), Asmussen (1995), and many others. A short review of the results is given in this section. The first-passage time problem of the Markov process (B, Z) is closely linked to the up-crossing and down-crossing ladder processes $\overline{B_t} := \max_{0\leq s\leq t} B_s$ and $\underline{B_t} := \min_{0\leq s\leq t} B_s$. The ladder processes observe B only when it is at its maximum or minimum, respectively. One can easily verify that $\overline{B_t}$ and $\underline{B_t}$ are again Markov processes on the same state space. Their generator matrices are linked to the so-called matrix Wiener–Hopf factorization (Q_+, Q_-) of the Markov process (B, Z), see Definition 1.

[2] An intensity matrix has negative diagonal and non-negative off-diagonal entries. Each row sums up to zero.

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