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On the maximum of a perturbed random walk

Q1 Alexander Iksanov [a,*], Andrey Pilipenko [b]

[a] *Faculty of Cybernetics, National Taras Shevchenko University of Kyiv, Kyiv, Ukraine*

[b] *Institute of Mathematics, National Academy of Sciences of Ukraine, Kyiv, Ukraine*

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ABSTRACT

Let $(\xi_1, \eta_1), (\xi_2, \eta_2), \ldots$ be a sequence of i.i.d. two-dimensional random vectors. We prove a functional limit theorem for the maximum of a perturbed random walk $\max_{0\le k\le n}(\xi_1+\cdots+\xi_k+\eta_{k+1})$ in a situation where its asymptotics is affected by both $\max_{0\le k\le n}(\xi_1+\cdots+\xi_k)$ and $\max_{1\le k\le n}\eta_k$ to a comparable extent. This solves an open problem that we learned from the paper "Renorming divergent perpetuities" by P. Hitczenko and J. Wesołowski.

1. Introduction and results

Let $(\xi_k, \eta_k)_{k\in\mathbb{N}}$ be a sequence of i.i.d. two-dimensional random vectors with generic copy (ξ, η). Let $(S_n)_{n\in\mathbb{N}_0}$ be the zero-delayed random walk with increments ξ_k for $k \in \mathbb{N}$, i.e.,

$$S_0 := 0 \quad \text{and} \quad S_n := \xi_1 + \cdots + \xi_n, \quad n \in \mathbb{N}.$$

Assuming that

$$\mathbb{E}\xi = 0 \quad \text{and} \quad v^2 := \operatorname{Var}\xi < \infty. \tag{1}$$

Hitczenko and Wesołowski in Hitczenko and Wesołowski (2011) investigated weak convergence of the one-dimensional distributions of $a_n \max_{0\le k\le n}(S_k+\eta_{k+1})$ as $n\to\infty$ for appropriate deterministic sequences (a_n). More precisely, in the proof of Theorem 3 in Hitczenko and Wesołowski (2011) it is shown that (I) whenever $\max_{0\le k\le n} S_k$ dominates $\max_{1\le k\le n+1}\eta_k$ the limit law of $a_n \max_{0\le k\le n}(S_k+\eta_{k+1})$ coincides with the limit law of $a_n \max_{0\le k\le n} S_k$ which is the law of $|B(1)|$ where $(B(t))_{t\ge 0}$ is a Brownian motion; and that (II) whenever $\max_{1\le k\le n+1}\eta_k$ dominates $\max_{0\le k\le n} S_k$ the limit law coincides with that of $a_n \max_{1\le k\le n+1}\eta_k$ which is a Fréchet law under a regular variation assumption.

If in addition to (1) condition

$$\mathbb{P}\{\eta > x\} \sim cx^{-2}, \quad x\to\infty \tag{2}$$

holds for some $c > 0$, then contributions of $\max_{0\le k\le n} S_k$ and $\max_{1\le k\le n+1}\eta_k$ to the asymptotic behavior of $\max_{0\le k\le n}(S_k+\eta_{k+1})$ are comparable. Hitczenko and Wesołowski conjectured (see Remark on p. 889 in Hitczenko and Wesołowski (2011)) that whenever conditions (1) and (2) hold, and ξ and η are independent, the limit random variable is $\theta + vB(1)$, where θ is independent of $B(1)$ and has a Fréchet distribution with parameters 2 and c. Under conditions (1) and (2) (not assuming that ξ and η are independent) we prove a functional limit result for $n^{-1/2}\max_{0\le k\le [n\cdot]}(S_k+\eta_{k+1})$ which implies that the conjecture is erroneous (see Remark 1.2).

* Corresponding author.
E-mail addresses: iksan@univ.kiev.ua, iksan72@mail.ru (A. Iksanov), pilipenko.ay@yandex.ua (A. Pilipenko).

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Denote by $D := D[0, \infty)$ the Skorokhod space of real-valued right-continuous functions which are defined on $[0, \infty)$ and have finite limits from the left at each positive point. Throughout the note we assume that D is equipped with the J_1-topology. For $c > 0$ defined in (2), let $N^{(c)} := \sum_k \varepsilon_{(t_k, j_k)}$ be a Poisson random measure on $[0, \infty) \times (0, \infty]$ with mean measure $\mathbb{LEB} \times \mu_c$, where $\varepsilon_{(t, x)}$ is the probability measure concentrated at $(t, x) \subset [0, \infty) \times (0, \infty]$, $\mathbb{LEB}$ is the Lebesgue measure on $[0, \infty)$, and μ_c is a measure on $(0, \infty]$ defined by

$$\mu_c\big((x, \infty]\big) = cx^{-2}, \quad x > 0.$$

Also, let $(B(t))_{t \geq 0}$ be a Brownian motion independent of $N^{(c)}$.

Theorem 1.1. *Suppose (1) and (2). Then*

$$n^{-1/2} \max_{0 \leq k \leq [n\cdot]} (S_k + \eta_{k+1}) \Rightarrow \sup_{t_k \leq \cdot} (vB(t_k) + j_k) \quad \text{as } n \to \infty$$

in D.

Remark 1.2. Observe that $\mathbb{P}\{\sup_{t_k \leq 1}(vB(t_k) + j_k) \geq 0\} = 1$, whereas $\mathbb{P}\{\theta + vB(1) < 0\} > 0$. This disproves the conjecture stated in Hitczenko and Wesołowski (2011). We note in passing that the law of $\sup_{t_k \leq 1}(vB(t_k) + j_k)$ is different from that of $\theta + v|B(1)| \overset{d}{=} \sup_{t_k \leq 1} j_k + v \sup_{t \leq 1} B(t)$, for $\sup_{t_k \leq 1}(vB(t_k) + j_k) < \sup_{t_k \leq 1} j_k + v \sup_{t \leq 1} B(t)$ a.s.

After the present note was ready for submission we learned that a version of Theorem 1.1, with ξ and η being independent, has also been proved, independently and at the same time, in Wang (2014) via a more complicated argument.

Let $C := C[0, \infty)$ be the set of continuous functions defined on $[0, \infty)$. Denote by M_p the set of Radon point measures ν on $[0, \infty) \times (-\infty, \infty]$ which satisfy

$$\nu([0, T] \times \{(-\infty, -\delta] \cup [\delta, \infty]\}) < \infty \tag{3}$$

for all $\delta > 0$ and all $T > 0$. The M_p is endowed with the vague topology. Define the functional F from $D \times M_p$ to D by

$$F(f, \nu)(t) := \begin{cases} \sup\limits_{k: \tau_k \leq t} (f(\tau_k) + y_k), & \text{if } \tau_k \leq t \text{ for some } k, \\ f(0), & \text{otherwise,} \end{cases}$$

where $\nu = \sum_k \varepsilon_{(\tau_k, y_k)}$. Assumption (3) ensures that $F(f, \nu) \in D$. If (3) does not hold, $F(f, \nu)$ may lost right-continuity.

Theorem 1.3. *For $n \in \mathbb{N}$, let $f_n \in D$ and $\nu_n \in M_p$. Assume that $f_0 \in C$ and*

- $\nu_0([0, \infty) \times (-\infty, 0]) = 0$ *and* $\nu_0(\{0\} \times (-\infty, +\infty]) = 0$,
- $\nu_0((a, b) \times (0, \infty]) \geq 1$ *for all positive a and b such that* $a < b$,
- $\nu_0 = \sum_k \varepsilon_{\left(\tau_k^{(0)}, y_k^{(0)}\right)}$ *does not have clustered jumps, i.e.,* $\tau_k^{(0)} \neq \tau_j^{(0)}$ *for* $k \neq j$.

If

$$\lim_{n \to \infty} f_n = f_0 \quad \text{in } D \tag{4}$$

and

$$\lim_{n \to \infty} \mathbb{1}_{[0, \infty) \times (0, \infty]} \nu_n = \nu_0 \quad \text{in } M_p, \tag{5}$$

then

$$\lim_{n \to \infty} F(f_n, \nu_n) = F(f_0, \nu_0) \tag{6}$$

in D.

Remark 1.4. The assumption $\nu_0((a, b) \times (0, \infty]) \geq 1$ for all positive $a < b$ is necessary. Indeed, set $\nu_n := \varepsilon_{(1, n^{-1})}$ for $n \in \mathbb{N}$, $\nu_0 := 0$ and $f_n(t) := t$ for $t \geq 0$ and $n \in \mathbb{N}_0$. Then, as $n \to \infty$,

$$F(f_n, \nu_n)(t) = (1 + n^{-1}) \mathbb{1}_{[1, \infty)}(t) \to \mathbb{1}_{[1, \infty)}(t) \neq F(f_0, \nu_0)(t) = 0.$$

The assumption $\nu_0((a, b) \times (0, \infty]) \geq 1$ for all positive $a < b$ can be omitted if one modifies the definition of F as follows:

$$F(f, \nu)(t) := \sup_{k: \tau_k \leq t} (f(\tau_k) + y_k) \vee \sup_{s \leq t} f(s).$$

Remark 1.5. Let $a > 0$ and $(T_n)_{n \in \mathbb{N}_0}$ be a random sequence independent of $(\eta_k)_{k \in \mathbb{N}}$. Further, denote by X a random process with a.s. continuous paths which is independent of (t_k^*, j_k^*) the atoms of a Poisson random measure on $[0, \infty) \times (0, \infty]$ with mean measure $\mathbb{LEB} \times \mu_{c, a}$, where $\mu_{c, a}$ is a measure on $(0, \infty]$ defined by $\mu_{c, a}\big((x, \infty]\big) = cx^{-a}$, $x > 0$. Whenever (2) holds

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