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Simulation of a strictly sub-Gaussian random field

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ABSTRACT

A model for simulation of a strictly sub-Gaussian random field is offered. An estimate for the rate of convergence of the model to the field is found.

1. Introduction

Numerical simulation of random processes is now quite a wide area. There exist many various methods for simulation of different types of stochastic processes (see e.g. Ogorodnikov and Prigarin, 1996; Ripley, 1987).

However, there is one substantial problem with many of these methods: as a rule, it is hard to assess the quality of approximation of a process by its model in terms of "distance" between paths of the process and the corresponding simulated paths. This is true, for instance, for spectral methods of simulation.

In this connection a problem of creating models which are free from this drawback deserves much interest. There exists an approach for building such models which is called simulation with given accuracy and reliability. Simulation with given accuracy and reliability is described in Kozachenko et al. (2007). This kind of simulation is considered, for example, also in Kozachenko and Pogoriliak (2011) and Kozachenko et al. (2005).

The essence of simulation with given accuracy and reliability can be described in the following way. A model (approximation) $\hat{X}(t)$ of a random process $X(t)$ is built, this model depends on certain parameters. Then the rate of convergence of the model to the process is described by a statement of the following type: if numbers ε (accuracy) and δ ($1-\delta$ is called reliability) are given and the parameters of the model satisfy certain restrictions (for instance, they are not less than certain lower bounds) then

$$P\{\|X-\hat{X}\|>\varepsilon\}\leq\delta. \tag{1}$$

There exist many such results for the cases when the norm in (1) is the L_p norm or the uniform norm.

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We study simulation of a strictly sub-Gaussian field (the notion of a strictly sub-Gaussian process is a generalization of the notion of a Gaussian process). We start from an expansion

$$X(t) = \sum_{i=1}^{n-1}\sum_{k=1}^{\infty}(d_{ik}(t)\xi_{ik} + e_{ik}(t)\zeta_{ik}) + \sum_{i=1}^{n-1} r_{i0}(t)\eta_{i0},$$

of a field $X(t)$, where ξ_{ik}, ζ_{ik}, η_{i0} are random variables, $d_{ik}(t)$, $e_{ik}(t)$, $r_{i0}(t)$ are functions which depend on $X(t)$ and a certain orthonormal basis. Our model is the partial sum

$$\hat{X}(t) = \hat{X}(k_1, k_2, \dots, k_{n-1}, t) = \sum_{i=1}^{n-1}\sum_{k=1}^{k_i-1}(d_{ik}(t)\xi_{ik} + e_{ik}(t)\zeta_{ik}) + \sum_{i=1}^{n-1} r_{i0}(t)\eta_{i0}.$$

We characterize the rate of convergence of $\hat{X}(t)$ to $X(t)$ in the following way (our approach is very close to simulation with given accuracy and reliability): if $\varepsilon > 0$ is given and the parameters $k_1, k_2, \dots, k_{n-1}$ are big enough then

$$\mathsf{E}\sup_{t\in\mathbf{T}}|X(t) - \hat{X}(t)| \le \varepsilon. \tag{2}$$

We can interpret (2) as follows: the average distance in the uniform norm between a path of the process and the corresponding path of the model is small enough. So simulated paths will be close to the corresponding paths of $X(t)$.

If the field $X(t)$ is Gaussian then the model $\hat{X}(t)$ can be used for computer simulation of $X(t)$. One of advantages of our model is the fact that the model of a Gaussian field is a Gaussian field. Besides, our method of simulation is very simple.

2. Auxiliary facts

A random variable ξ is called *sub-Gaussian* if there exists such a constant $a \ge 0$ that $\mathsf{E}\exp\{\lambda\xi\} \le \exp\{\lambda^2 a^2/2\}$ for all $\lambda \in \mathbb{R}$ and *strictly sub-Gaussian* if, in addition, $\inf\{a \ge 0 : \mathsf{E}\exp\{\lambda\xi\} \le \exp\{\lambda^2 a^2/2\}, \lambda \in \mathbb{R}\} = (\mathsf{E}\xi^2)^{1/2}$. The class of all sub-Gaussian random variables on a standard probability space $\{\Omega, \mathcal{B}, P\}$ is a Banach space with respect to the norm

$$\tau(\xi) = \inf\{a \ge 0 : \mathsf{E}\exp\{\lambda\xi\} \le \exp\{\lambda^2 a^2/2\}, \lambda \in \mathbb{R}\}.$$

Two examples of sub-Gaussian random variables are a centered Gaussian random variable and a random variable uniformly distributed on $[-b, b]$.

A family Δ of sub-Gaussian random variables is called *strictly sub-Gaussian* if for any finite or countable set I of random variables $\xi_i \in \Delta$ and for any $\lambda_i \in \mathbb{R}$

$$\tau^2\left(\sum_{i\in I}\lambda_i\xi_i\right) = \mathsf{E}\left(\sum_{i\in I}\lambda_i\xi_i\right)^2.$$

A stochastic process $X = \{X(t), t \in \mathbf{T}\}$ is called *sub-Gaussian* if all the random variables $X(t)$, $t \in \mathbf{T}$, are sub-Gaussian. We call a stochastic process $X = \{X(t), t \in \mathbf{T}\}$ *strictly sub-Gaussian* if the family $\{X(t), t \in \mathbf{T}\}$ is strictly sub-Gaussian. Any centered Gaussian process is strictly sub-Gaussian.

Details about sub-Gaussian random variables and processes can be found in Buldygin and Kozachenko (2000).

We consider a partition of an interval $[b_1, b_n] \subset \mathbb{R}$: $b_1 < b_2 < \cdots < b_n$ and an orthonormal basis $\mathcal{F}$ in $L_2([b_1, b_n])$ defined as

$$\mathcal{F} = \bigcup_{i=1}^{n-1}\mathcal{F}_i, \tag{3}$$

where

$$\mathcal{F}_i = \left\{\frac{1}{\sqrt{2\Delta_i}}, \frac{1}{\sqrt{\Delta_i}}\cos\frac{k\pi(x - b_i')}{\Delta_i}, \frac{1}{\sqrt{\Delta_i}}\sin\frac{k\pi(x - b_i')}{\Delta_i}, k = 1, 2, \dots\right\},$$
$$\Delta_i = (b_{i+1} - b_i)/2, \qquad b_i' = (b_{i+1} + b_i)/2.$$

We will need the following simple fact (which is a corollary from Theorem 1.1 in Kozachenko et al. (2011)).

Theorem 1. *A centered second-order random field $X = \{X(t), t \in \mathbf{T}\}$, $\mathbf{T} \subset \mathbb{R}^m$, with the correlation function*

$$R(t, s) = \int_{b_1}^{b_n} u(t, y)\overline{u(s, y)}dy, \tag{4}$$

where $u(t, \cdot) \in L_2([b_1, b_n])$, can be represented as a mean square convergent series

$$X(t) = \sum_{i=1}^{n-1}\sum_{k=1}^{\infty}(d_{ik}(t)\xi_{ik} + e_{ik}(t)\zeta_{ik}) + \sum_{i=1}^{n-1} r_{i0}(t)\eta_{i0}, \tag{5}$$

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