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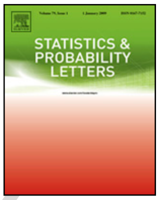

Smooth density for the solution of scalar SDEs with locally Lipschitz coefficients under Hörmander condition

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In this paper the existence of a smooth density is proved for the solution of an SDE, with locally Lipschitz coefficients and semi-monotone drift, under Hörmander condition. We prove the nondegeneracy condition for the solution of the SDE, from it an integration by parts formula would result in the Wiener space. To this end we construct a sequence of SDEs with globally Lipschitz coefficients whose solutions converge to the original one and use some Lyapunov functions to show the uniform boundedness of the p-moments of the solutions and their Malliavin derivatives with respect to n.

1. Introduction

In this paper, we study Malliavin differentiability and the existence of a smooth density for the solution of an SDE. We consider a scalar SDE whose semi-monotone drift and locally Lipschitz coefficients satisfy the Hörmander condition. Such equations are considered mostly in finance, biology, and dynamical systems and are more challenging when considered on infinite dimensional spaces (see e.g. Bahlali (1999), Zangeneh (1995) and Gyöngy and Millet (2007)).

We prove the existence of a unique, infinitely Malliavin differentiable, strong solution to this SDE satisfying some nondegeneracy condition, and derive both the integration by parts formula in the Wiener space and the existence of a smooth density for this solution.

This subject has been studied by many authors, mostly in the case where the coefficients are globally Lipschitz. Kusuoka and Stroock (1984) have shown that an SDE whose coefficients are C^∞-globally Lipschitz with polynomial growth, has a strong Malliavin differentiable solution of any order. The absolute continuity of the law of the solution of SDEs with respect to the Lebesgue measure and the smoothness of its density under some nondegeneracy condition are shown in Nualart (2006) and Bichteler et al. (1987). Nualart (2006) shows that the Hörmander condition, posed on the coefficients, condition (H) in this paper, implies this required nondegeneracy condition, if the coefficients are C^∞-globally Lipschitz and have polynomial growth. Assuming the nondegeneracy condition one can derive some integration by parts formula in the Wiener space and also regularity for the distribution of the solution (see e.g. Nualart (2006)). It is often of interest for investors to

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derive an option pricing formula and to know its sensitivity with respect to various parameters. The integration by parts formula obtained from Malliavin calculus can transform the derivative of the option price into a weighted integral of random variables. This gives much more accurate and fast converging numerical solutions than obtained by the classical methods (Kohatsu-Higa and Montero, 2004; Bavouzet and Messaoud, 2006). The interested reader could see Alòs and Ewald (2008) and Marco (2011).

In recent years, there were attempts to generalize these results to SDEs with non-globally Lipschitz coefficients. For example, in Fournier and Printems (2010) using a Fourier transform argument, some absolute continuity results are obtained for the law of the solution of an SDE with Hölder coefficients. The existence of densities for a general class of non-Markov Itö processes under some spatial ellipticity condition and that allow the degeneracy of the diffusion coefficient is shown in Bell (2004). Marco (2011) has shown that assuming some local properties of coefficients, and uniform ellipticity of the diffusion coefficient, the law of the solution of the SDE has smooth density. If the diffusion coefficient is uniformly elliptic, then the Hörmander condition is satisfied. When the noise is a fractional Brownian motion or a Levy process the same results are obtained under ellipticity and Hörmander condition as well. For other references on this subject, we refer the reader to Kusuoka (2010), Marco (2010), Hiraba (1992) and Baudoin et al. (submitted for publication).

To deal with the SDE with non-globally Lipschitz coefficients, we construct a sequence of SDEs with globally Lipschitz coefficients whose solutions are Malliavin differentiable of any order and satisfy a nondegeneracy condition. In this way we can apply the classical Malliavin calculus to the solutions. We can find also a uniform bound for the moments of the solutions, and all their Malliavin derivatives, by using some Lyapunov functions. Then we will prove the nondegeneracy condition for the original SDE, using the nondegeneracy condition for the sequence of solutions to the constructed SDEs. This result implies the integration by parts formula in the Wiener space and the existence of the smooth density for the solution.

The paper is organized as follows. In Section 2, we recall some basic results from Malliavin calculus that will be used in the paper, in particular the integration by parts formula due to Nualart (2006, Proposition 2.1.4). In Section 3, we state the assumptions and our main results. In Section 4, we prove the uniformly boundedness for the moments of Malliavin derivatives of the solution to a sequence of approximating SDEs, as there exist some Lyapunov functions. Section 5 involve the construction of our approximating SDEs with globally Lipschitz coefficients, and proving the convergence of their solutions and their Malliavin derivatives to those of the solution of the original SDE (3.1). Also, we introduce some Lyapunov functions which results in the infinitely weak differentiability. In Section 6, we will prove the nondegeneracy condition that implies the integration by parts formula and the existence of smooth density.

2. Some basic results from Malliavin calculus

In this article, we use the same notations as in Nualart (2006). Let Ω denote the Wiener space $C_0([0, T]; \mathbb{R})$ endowed with $\|\ .\ \|_\infty$-norm making it a (separable) Banach space. Consider a complete probability space $(\Omega, \mathcal{F}, P)$, in which $\mathcal{F}$ is generated by the open subsets of the Banach space Ω, W_t is a d-dimensional Brownian motion, and $\mathcal{F}_t$ is the filtration generated by W_t.

Consider the Hilbert space $H := L^2([0, T]; \mathbb{R})$. The Malliavin derivative operator D is closable from $L^p(\Omega)$ to $L^p(\Omega, H)$, for every $p \geq 1$ and the adjoint of the operator D is denoted by δ. We use the notation $\bigwedge_F = \|DF\|_H^2$ to show the Malliavin covariance matrix for a random variable F, and for every $k \geq 1$, we set $D_{r_1\cdots r_k}F = D_{r_k}(D_{r_1\cdots r_{k-1}}F)$.

Now let Y_t be a solution to the following SDE;

$$dY_t = B(Y_t)dt + A(Y_t)dW_t \qquad Y_0 = x_0, \tag{2.1}$$

where $B : \mathbb{R} \longrightarrow \mathbb{R}$ is a measurable function and $A : \mathbb{R} \longrightarrow \mathbb{R}$ is an C^∞ function. Let Z_t be the solution of the following linear SDE;

$$Z_t = 1 + \int_0^t B'(Y_s)Z_s ds + \int_0^t A'(Y_s)Z_s dW_s$$

and

$$C_t := \int_0^t (Z_s^{-1}A(Y_s))^2 ds,$$

and assume the Hörmander's condition holds as follows:

(H) $A(x_0) \neq 0$ or $A^{(n)}(x_0)B(x_0) \neq 0$ for some $n \geq 1$.

Under this condition Nualart (2006) has shown the following proposition.

Proposition 2.1. *For a solution Y_t to an SDE with globally Lipschitz coefficients and polynomial growth for all their derivatives, the Hörmander's condition* (H) *implies that for any $p \geq 2$ and any ϵ small enough,*

$$P\Big(C_t \leq \epsilon\Big) \leq \epsilon^p \tag{2.2}$$

and $(\det C_t)^{-1} \in L_p(\Omega)$ *for all p. Thereby obtaining the nondegeneracy condition for Y_t and thus the integration by parts formula in the Wiener space and an infinitely differentiable density, too.*