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# Sign tests using ranked set sampling with unequal set sizes

Liangyong Zhang [a,b,*], Xiaofang Dong [a], Xingzhong Xu [b]

[a] *School of Science, Qiqihar University, Qiqihar, China*
[b] *School of Mathematics, Beijing Institute of Technology, Beijing, China*

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ABSTRACT

This paper considers sign test under ranked set sampling with unequal set sizes (RSSU), and proposes weighted sign tests associated with judgment ranks. The optimal weight vector is shown to be distribution-free, and RSSU is shown to be more efficient than ranked set sampling.



## 1. Introduction

Ranked set sampling (RSS) introduced by McIntyre (1952) is a sampling protocol that can often be used to improve the cost efficiency of an experiment. It is often used when a ranking of the sampling units can be obtained cheaply without having to actually measure the characteristic of interest, which may be time consuming or costly. For more details see Chen et al. (2004) and references therein.

In a RSS procedure, one first draw $m$ random samples with $m$ units in each sample. The units in each sample are ranked by some auxiliary criterion that does not require actual measurements and only the h smallest observation is quantified from the h sample, $i = 1, \ldots, m$. This completes a cycle of the sampling, the cycle is repeated $k$ times to obtain a ranked set sample of size $n = mk$. Such a sampling technique is well received and widely applicable in environment, economy, reliability, etc., (Nussbaum and Sinha, 1997; Mode et al., 1999; Wolfe, 2004; Bocci et al., 2010; Dong et al., 2012).

Recently, interest has been shown in sign test using RSS by several investigators. Hettmansperger (1995) first considered the sign test based on RSS data and found that the RSS sign test is more efficient than the sign test using simple random sampling (SRS). Ozturk (1999) and Ozturk and Wolfe (2000) carried out extensive numerical studies and suggested the median ranked set sampling (MRSS) selecting the median observations for quantification to test median. Wang and Zhu (2005) discussed the sign test for median using unbalanced RSS and showed analytically that the sampling allocation that maximizes the efficacy is MRSS. Kaur et al. (2002) and Dong and Cui (2010) extended sign tests for population quantiles using unbalanced RSS and identified the optimal allocation, finding that it depends on the quantile but not on the parent population. In addition, Wang and Zhu (2005) and Dong and Cui (2010) both proposed weighted sign tests under unbalanced RSS, and proved the weighted version always improves the Pitman efficiency for all distributions.

Bhoj (2001) suggested using the ranked set sampling with unequal set sizes (RSSU) method to estimate the population mean, and showed that the estimator based on RSSU are more efficient than the estimators based on SRS and RSS in some

* Corresponding author at: School of Science, Qiqihar University, Qiqihar, China.
*E-mail address:* ly1980z@126.com (L. Zhang).

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cases. Dong et al. (2013) used RSSU to estimate the reliability $P(X > Y)$ for a system with strength $X$ and stress $Y$ both following exponential distributions. The best linear unbiased estimator and the modified maximum likelihood estimator based on RSSU were shown to be superior to the known estimators based on SRS and RSS. The sampling procedure for RSSU involves first drawing $m$ samples, where the size of the h sample is $m_i = 2i - 1$, $i = 1, \ldots, m$. Obviously, when $m$ is even, the half of the sample sizes are smaller than $m$ and the other half is greater than $m$. In the case of odd $m$, one sample is of size $m$, $(m-1)/2$ samples are greater than $m$, and $(m-1)/2$ samples are smaller than $m$. Then, in the h sample the observation having the h judgment rank is quantified for $i = 1, \ldots, m$. The entire process is repeated $k$ time to obtain a RSSU of size. It is easy to show that $\sum_{i=1}^{m} m_i = m^2$. In each cycle, RSS and RSSU schemes both measure accurately only $m$ observations out of $m^2$ ranked units. Hence, the comparison of the tests based on RSS and RSSU will be fair.

In this paper, we will consider the one sample sign test based on RSSU. In Section 2, we give the sign test statistic under RSSU, and the exact null distribution of the test statistic is established. It is shown that the small sample power and the large sample efficiency of the RSSU sign test are higher than that of the RSS sign test. In Section 3, because observations with different ranks are not identically distributed, we give the weighted sign test in which weights associate with the judgment rank, and present the asymptotic properties of the weighted sign test. The optimal weight vector is identified and shown to be distribution free. Then, it is shown that the optimal weighted sign test under RSSU is not only superior to the optimal weighted sign test under RSS but also more efficient than the sign test under MRSS. Section 4 discusses the case of imperfect ranking. Finally, Section 5 provides some concluding remarks.

## 2. Sign test using RSSU

Let be a known constant, the null hypothesis asserts that $\theta$ is the median of infinite population having cumulative distribution function (CDF) $F(x)$ and probability density function (PDF) $f(x)$. Thus, we want to test $H_0 : F(\theta) = 1/2$ against either a one-sided or a two-sided alternative.

### *2.1. Sign test statistic using RSSU and its properties*

Let $X_{(i:m_i)j}$, $i = 1, \ldots, m$ and $j = 1, \ldots, k$ be a RSSU of size $n = mk$ drawn from $F$, where $X_{(i:m_i)j}$ is the $j$th observation on the $i$th order statistic of a SRS of size $m_i = 2i - 1$. The distribution of $X_{(i:m_i)j}$, which depends on the rank order $i$ but not on $j$, has PDF and CDF as follows,

$$f_{(i:m_i)}(x) = \frac{1}{B(i, i)}[F(x) - F^2(x)]^{i-1} f(x), \tag{1}$$

$$F_{(i:m_i)}(x) = \frac{B(F(x); i, i)}{B(i, i)}, \tag{2}$$

where $B(t; a, b) = \int_0^t u^{a-1}(1-u)^{b-1} du$ and $B(a, b) = B(b, a) = B(1, a, b)$.

The sign test statistic under RSSU for population median is given by

$$S^+_{RSSU} = \sum_{i=1}^{m}\sum_{j=1}^{k} \psi\{X_{(i:m_i)j} - \theta\} = \sum_{i=1}^{m} \eta_i, \tag{3}$$

where $\psi(t)$ is the indicator function $I(t > 0)$ and

$$\eta_i = \sum_{j=1}^{k} \psi\{X_{(i:m_i)j} - \theta\} \sim \text{binomial}(k, 1 - F_{(i:m_i)}(\theta)). \tag{4}$$

It is easy to verify that

$$E(S^+_{RSSU}) = k\sum_{i=1}^{m}[1 - F_{(i:m_i)}(\theta)], \tag{5}$$

$$Var(S^+_{RSSU}) = k\sum_{i=1}^{m} F_{(i:m_i)}(\theta)[1 - F_{(i:m_i)}(\theta)]. \tag{6}$$

Under $H_0$, $F(\theta) = 1/2$, from (2) we get $F_{(i:m_i)}(\theta)|_{H_0} = 1/2$. Thus, $\eta_i \sim \text{binomial}(k, 1/2)$, $E(S^+_{RSSU})|_{H_0} = \frac{n}{2}$ and $Var(S^+_{RSSU})|_{H_0} = \frac{n}{4}$. Since $S^+_{RSSU}$ is a sum of binomial random variables with same parameters, we get the following theorem.

**Theorem 1.** *For fixed n and under* $H_0$, $S^+_{RSSU} \sim \text{binomial}(n, 1/2)$.

The above theorem allows us to carry out a size $\alpha$ test. For example, with $\alpha = 0.05$ we would reject $H_0 : F(\theta) = 1/2$ in favor of $H_1 : F(\theta) > 1/2$ if $S^+_{RSSU} \leq 4$ when $n = 15$. However, the distribution of $S^+_{RSSU}$ does not have a simple closed form under $H_1$. The following theorem gives the exact distribution of $S^+_{RSSU}$ in general cases.