Statistics and Probability Letters 85 (2014) 98–105

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

On the Markovian projection in the Brunick–Shreve mimicking result

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ARTICLE INFO

Article history:
Received 20 June 2013
Received in revised form 11 November 2013
Accepted 15 November 2013
Available online 21 November 2013

Keywords:
Mimicking
Diffusion process
Local volatility models
Markovian projection

ABSTRACT

For a one-dimensional Itô process $X_t = \int_0^t \sigma_s dW_s$ and a general $\mathcal{F}_t^X$-adapted non-decreasing path-dependent functional Y_t, we derive a number of forward equations for the characteristic function of (X_t, Y_t) for absolutely and non absolutely continuous functionals Y_t. The functional Y_t can be the maximum, the minimum, the local time, the quadratic variation, the occupation time or a general additive functional of X. Inverting the forward equation, we obtain a new Fourier-based method for computing the Markovian projection $\mathbb{E}(\sigma_t^2|X_t, Y_t)$ explicitly from the marginals of (X_t, Y_t), which can be viewed as a natural extension of the Dupire formula for local volatility models; $\mathbb{E}(\sigma_t^2|X_t, Y_t)$ is a fundamental quantity in the important mimicking theorems in Brunick and Shreve (2013). We also establish mimicking theorems for the case when Y is the local time or the quadratic variation of X (which is not covered by Brunick and Shreve (2013)), and we derive similar results for trivariate Markovian projections.

1. Introduction

There has been a growing literature on the problem of constructing a process that mimics certain properties of a given Itô process, but is simpler in the sense that the mimicking process solves a stochastic differential equation, or more generally a stochastic functional differential equation, while the original Itô process may have drift and diffusion terms that are themselves adapted stochastic processes. The classical paper of Gyöngy (1986) considers a multi-dimensional Itô process, and constructs a weak solution to a stochastic differential equation which mimics the marginals of the original Itô process at each fixed time. The drift and covariance coefficient for the mimicking process can be interpreted as the expected value of the instantaneous drift and covariance of the original Itô process, conditioned on its terminal level.

Brunick and Shreve (2013) relax the conditions of non-degeneracy and boundedness on the covariance of the Itô process imposed in Gyöngy (1986), and they also significantly extend the Gyöngy result. More specifically, the main result Theorem 3.5 in Brunick and Shreve (2013) proves that we can match the *joint* distribution at each fixed time of various functionals of the Itō process, including the maximum-to-date or the running average of one component of the Itô process. The mimicking process now takes the form of a stochastic functional differential equation (SFDE) and the diffusion coefficient for the SFDE is given by the so-called *Markovian projection*; in the case when we are mimicking the law of the terminal value of the process X_t and another path-dependent functional Y_t, the Markovian projection is given by the conditional expectation $\hat{\sigma}(x, y, t)^2 = \mathbb{E}(\sigma_t^2 \mid X_t = x, Y_t = y)$.

Brunick and Shreve (2013) do not provide a constructive method for computing $\hat{\sigma}(x, y, t)^2$; however, for the standard problem of just mimicking the law of the terminal value of the process, this can be computed from the well known Dupire forward equation for continuous semimartingales, in terms of infinitesimal calendar and butterfly spreads of put or call options. This equation was derived heuristically in Dupire (2004) and can be proved rigorously using the Tanaka–Meyer

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0167-7152/$ – see front matter © 2014 Published by Elsevier B.V.
http://dx.doi.org/10.1016/j.spl.2013.11.005

formula for continuous semimartingales, see Klebaner (2002). Bentata and Cont (2009) extend this analysis to derive a forward partial integro-differential equation for the call option price (in the sense of distributions) when the underlying asset follows a (possibly) discontinuous semimartingale. In another article, Bentata and Cont (in preparation) have also extended the Gyöngy mimicking result to jump diffusion processes, but they assume a priori that the Markovian projection is continuous; it is not clear if/when this holds if we do not also assume a priori that the original Itô process admits a positive density at the point (x, y) of interest.

The other main technical obstacle in establishing fitting and mimicking results of this nature is establishing *uniqueness* for the associated forward Kolmogorov equation (or associated partial integro-differential equation when there is a jump component), in the sense of distributions. This can be done when Y_t is an a.s. absolutely continuous functional using standard existence and uniqueness theorems for the forward Kolmogorov equation associated with the mimicking diffusion process, which is degenerate when we are just mimicking the marginals of the two quantities (X_t, Y_t), because there is only one driving Brownian motion. It is less clear how to proceed for a.s. non-absolutely continuous functionals like the running maximum or local time, because the mimicking process now takes the form of a non-standard stochastic functional differential equation for which the theory is less developed.

In this article, we consider an $\mathbb{R}$-valued square integrable Itô semimartingale of the form $dX_t = \sigma_t dW_t$, and a general $\mathcal{F}_t^X$-adapted non-decreasing process Y_t (this is our path-dependent functional of interest). We first consider the case when Y_t is a.s. non absolutely continuous and $X_t = g(Y_t)$ for some continuous function $g(.)$, on the *growth set* of Y_t; this condition is satisfied when for example when Y is the running maximum of X with $g(y) = y$, or if $Y_t = L_t^a$ the local time of X at a with $g(y) = a$. In this setup, we derive a general forward equation for the Fourier–Laplace transform of the law of (X_t, Y_t), and the forward equation can be inverted to compute the Markovian projection $\hat{\sigma}(x, y, t)^2$ explicitly via a Fourier–Laplace inversion, without the a priori assumption that (X_t, Y_t) has a density at (x, y) or that $\hat{\sigma}(t, ., .)$ is continuous at (x, y). In Section 5, we consider the case when Y_t is an a.s. absolutely continuous functional; we first derive a mimicking result for the case when $Y_t = \langle X \rangle_t$ which is not considered in Brunick and Shreve (2013), and we then derive a forward equation for the Fourier–Laplace transform of the law of (X_t, Y_t) and a similar equation when Y is an additive functional of X. In both cases, we can use the forward equation to compute the appropriate Markovian projection, and we conclude the article with a similar forward equation for a trivariate Markovian projection.

2. The modelling setup

We let $X : [0, T] \times \Omega \mapsto \mathbb{R}$ denote an Itô process, i.e. a continuous martingale defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ satisfying the usual conditions, with stochastic integral representation

$$X_t = \int_0^t \sigma_s dW_s, \tag{2.1}$$

where W is a standard one-dimensional Brownian motion adapted to $\mathcal{F}_t$, and σ_t is an adapted process with $\mathbb{E}(\int_0^t \sigma_s^2 ds) < \infty$ for all $t \le T$. Let $\mathcal{F}_t^X$ denote the natural filtration of X. Throughout, we let $(Y_t)_{t\ge 0}$ denote an a.s. continuous non-decreasing $\mathcal{F}_t^X$-adapted process with $Y_0 = 0$—this is our path-dependent functional of interest. We assume that Y_t has full support on $\mathbb{R}^+$ and that $\mathbb{E}(Y_t) < \infty$ for all $t \le T$.

We begin with a short technical lemma.

Lemma 2.1. *There exists a function $\hat{\sigma}^2 : \mathbb{R} \times \mathbb{R}^+ \times (0, T] \mapsto \mathbb{R}^+$ such that $\hat{\sigma}^2(., ., t)$ is (Borel) measurable and*

$$\mathbb{E}(\sigma_t^2 \mid X_t, Y_t) = \hat{\sigma}^2(X_t, Y_t, t) \quad \text{a.s.}$$

Proof. See Appendix C. □

Remark 2.2. We refer to $\hat{\sigma}^2(x, y, t)$ as the Markovian projection of σ_t^2 on (X_t, Y_t).

2.1. The Brunick–Shreve mimicking result

We now briefly summarize the main result in Brunick and Shreve (2013) for the special case when the dimension $n = 1$ and the process under consideration is driftless.

For an Itô process of the form in (2.1), Brunick and Shreve (2013) consider a certain class of path-dependent functionals Y of X (which they refer to as *updating functions*), which can include $Y_t = \bar{X}_t$ (the running maximum of X), $Y_t = \underline{X}_t$ (the running minimum of X) or an additive functional of the form $Y_t = \int_0^t g(X_s)ds$, but cannot include $\langle X \rangle_t$ the quadratic variation of X or L_t^a the local time of X at $x = a$ because these functionals are not continuous in the sup norm topology. For continuous functionals Y in the class of updating functions, the main Theorem 3.6 in Brunick and Shreve (2013) proves that there exists a filtered probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathcal{F}}_t, \hat{\mathbb{P}})$ that supports a continuous adapted process $\hat{X}$ on $\mathbb{R}$ and a one-dimensional Brownian motion $\hat{W}$ satisfying

$$\hat{X}_t = \int_0^t \hat{\sigma}(\hat{X}_s, \hat{Y}_s, s) d\hat{W}_s,$$