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On the reversed hazard rate of sequential order statistics

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ABSTRACT

Sequential order statistics can be used to describe the lifetime of a system with n components which works as long as k components function assuming that failures possibly affect the lifetimes of remaining units. In this work, the reversed hazard rates of sequential order statistics are examined. Conditions for the reversed hazard rate ordering and the decreasing reversed hazard rate property of sequential order statistics are given.

1. Introduction

Kamps (1995a) introduced the concept of sequential order statistics (SOS) as an extension of the order statistics (OS) model. Following Cramer and Kamps (2003), sequential order statistics can be defined as follows: let $F_1, \ldots, F_n$ be continuous distribution functions with $F_1^{-1}(1) \leq \cdots \leq F_n^{-1}(1)$ and let $B_1, \ldots, B_n$ be independent random variables where B_i is beta distributed with parameters $n-i+1$ and 1, $1 \leq i \leq n$. Then the random variables

$$X_{i:n}^* = F_i^{-1}\left(1 - B_i \overline{F}_i(X_{i-1:n}^*)\right), \quad \text{for } i = 1, \ldots, n,$$

are called sequential order statistics.

Note that OS are contained in the model of SOS via the specific choice $F_1 = \cdots = F_n$. In the reliability context, there exists a relation between SOS and the lifetimes of sequential k-out-of-n systems, in the same way that there exists a connection between OS and the lifetimes of k-out-of-n systems. In this case, the $(n-k+1)$th SOS in a sample of size n represents the lifetime of a sequential k-out-of-n system (see Cramer and Kamps, 2001). A sequential k-out-of-n system is more flexible than a k-out-of-n system in the sense that, after the failure of some component, the distribution of the residual lifetime of the components at work may change.

The model of SOS is closely connected to several other models of ordered random variables. For instance, it is well known that the specific choice of distribution functions $F_i(t) = 1 - (1 - F(t))^{\alpha_i}$, $t \in \mathbb{R}$, $1 \leq i \leq n$, with a continuous distribution function F and positive real numbers $\alpha_1, \ldots, \alpha_n$ leads to the model of generalised order statistics with parameters $\gamma_i =$

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$(n-i+1)\alpha_i$, $1 \le i \le n$. Further results about SOS and related models can be found, for instance, in Kamps (1995b), Cramer and Kamps (1996), Kamps and Cramer (2001), Cramer (2006), Balakrishnan et al. (2008), Beutner (2008, 2010), Beutner and Kamps (2009), Burkschat (2009), Bedbur (2010), Burkschat et al. (2010), Balakrishnan et al. (2011) and Bedbur et al. (2012).

In this article, we focus on particular stochastic comparisons and ageing properties of SOS. Some recent articles on these subjects are, e.g., Zhuang and Hu (2007), Burkschat and Navarro (2011), Navarro and Burkschat (2011) and Torrado et al. (2012). We will present some results on the reversed hazard rate ordering and its associated ageing notion, the decreasing reversed hazard rate (DRHR) property (see, e.g., Block et al., 1998; Sengupta and Nanda, 1999; Chandra and Roy, 2001; Nanda and Shaked, 2001; Finkelstein, 2002; Nanda et al., 2003; Ahmad and Kayid, 2005; Marshall and Olkin, 2007; Shaked and Shanthikumar, 2007). Recent results on the DRHR property of some ordered random variables are given in Kundu et al. (2009) and Wang and Zhao (2010).

In Section 2, we recall the definitions of the reversed hazard rate ordering, the DRHR property, and give some notations for SOS. The main results are given in Sections 3 and 4. More precisely, we investigate conditions on the underlying distribution functions on which the SOS are based, in order to compare SOS in the reversed hazard rate ordering and to obtain the DRHR property of SOS.

Throughout the article we use the terms increasing and decreasing in the weak sense, that is, a function g is called increasing (decreasing) if $x \le y$ implies $g(x) \le (\ge) g(y)$. Furthermore, we assume that the distributions of the occurring random variables have the same support which is given by an interval of the real line.

2. Definitions and notations

Let X be a non-negative random variable describing a lifetime with the distribution function F, survival function $\bar{F} = 1-F$, density function f and reversed hazard rate function $r_X = f/F$. Analogously, let Y be a non-negative random variable with the distribution function G, survival function $\bar{G} = 1 - G$, density function g and reversed hazard rate function $r_Y = g/G$. First, we recall the definition of the reversed hazard rate order (see, e.g., Shaked and Shanthikumar, 2007, Section 1.B.6).

Definition 1. The random variable X is said to be smaller than Y in the reversed hazard rate order (denoted by $X \le_{\mathrm{rh}} Y$) if $r_X(t) \le r_Y(t)$ for all $t \ge 0$.

Let the random variable $X_{(t)}$ be distributed as the time elapsed since the failure time X of a unit, given that the unit failed at or before time $t > 0$, i.e., let the distribution theoretical identity

$$X_{(t)} \stackrel{st}{=} [t - X \mid X \le t]$$

hold. The random variable $X_{(t)}$ is known as the inactivity time or the reversed residual life of X at time t. Its survival function is given by

$$P\left(X_{(t)} > x\right) = \frac{F(t-x)}{F(t)}, \quad 0 \le x < t.$$

The reversed hazard rate ordering is related to the random variable $X_{(t)}$, since $X \le_{\mathrm{rh}} Y$ if $X_{(t)} \ge_{\mathrm{st}} Y_{(t)}$ for all $t \ge 0$ (see Shaked and Shanthikumar, 2007, Section 1.B.6).

Related to this ordering, the decreasing reversed hazard rate (DRHR) class of life distributions has been introduced and studied in the literature (see, e.g., Sengupta and Nanda, 1999).

Definition 2. The random variable X is said to have a decreasing reversed hazard rate (denoted by DRHR) if $r_X(t)$ is decreasing in t.

Finally, we recall some results from the distribution theory of SOS. Let $X^*_{1:n}, \ldots, X^*_{n:n}$ be the SOS based on distribution functions $F_1, \ldots, F_n$ with respective density functions $f_1, \ldots, f_n$. Let $h_i = f_i/\overline{F}_i$, $i = 1, \ldots, n$, denote the hazard rates. Based on the results in Cramer and Kamps (2003), we can assume that

$$X^*_{1:n} = H_1^{-1}(Z_1),$$
$$X^*_{i:n} = H_i^{-1}(Z_i + H_i(X^*_{i-1:n})), \quad \text{for } i = 2, 3, \ldots, n,$$

where H_i^{-1} denotes the inverse function of the cumulative hazard function $H_i = -\ln \overline{F}_i$ and $Z_1, \ldots, Z_n$ are independent random variables where Z_i is exponential distributed with parameter $n-i+1$, $1 \le i \le n$. The density function of the first SOS is given by

$$f_{*,1}(t) = nh_1(t)\bar{F}_{*,1}(t),$$

with the reversed hazard rate

$$r_{*,1}(t) = nh_1(t)\frac{\bar{F}_{*,1}(t)}{F_{*,1}(t)}.$$