Statistics and Probability Letters 85 (2014) 144–152

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Monotonicity of certain integrals involving gamma distributions and their applications in multiple comparisons

Neeraj Misra, Mohd. Arshad *

Department of Mathematics and Statistics, Indian Institute of Technology, Kanpur, Kanpur-208016, India

ARTICLE INFO

Article history:
Received 11 September 2013
Received in revised form 21 November 2013
Accepted 22 November 2013
Available online 1 December 2013

Keywords:
Monotonicity
Multiple comparisons
Ranking and selection

ABSTRACT

Barlow and Gupta (1969) and Alam (1970) studied the monotonicity of two integrals, involving gamma distributions, that arise in certain ranking and selection problems. In this paper, we shall unify their results by studying the monotonicity of two generalized versions of integrals considered by them. We will also provide applications of derived results in study of certain multiple comparison procedures.

1. Introduction and formulation of the problem

Let $Z_1, \ldots, Z_k$ be $k\,(\geq 2)$ independent and identically distributed (i.i.d.) gamma random variables, each having the cumulative distribution function (c.d.f.)

$$G_\alpha(x) = \begin{cases} 0, & \text{if } x < 0 \\ \displaystyle\int_0^x \frac{1}{\Gamma(\alpha)} z^{\alpha-1} e^{-z}\, dz, & \text{if } x \geq 0, \end{cases} \tag{1.1}$$

where $\alpha > 0$ is the shape parameter and $\Gamma(\cdot)$ is the usual gamma function. Integrals involving gamma distributions arise in many statistical inference problems. Gupta (1963) proposed a selection procedure for the problem of selecting a subset of k gamma populations which contains the best population (a population having the largest scale parameter). The expression for infimum of probability of correct selection of the proposed selection procedure involves the integral

$$I_1(\alpha) = \int_0^\infty \left(G_\alpha\left(\frac{t}{c}\right)\right)^{k-1} dG_\alpha(t), \quad \alpha > 0, \tag{1.2}$$

where $c \in (0, 1)$ and $k \in \{2, 3, \ldots\}$ are fixed constants. From the numerical study reported in Gupta (1963) it can be observed that, for fixed k and c, the integral $I_1(\alpha)$ is increasing in $\alpha \in \{1, 2, \ldots\}$. It follows from van Zwet (1964) that the family $\{G_\alpha(\cdot), \alpha > 0\}$ of gamma distributions is convex ordered, that is, for $0 < \beta_1 < \beta_2$, the function $G_{\beta_1}^{-1}(G_{\beta_2}(x))$ is convex on $(0, \infty)$. This in turn implies that the function $G_{\beta_1}^{-1}(G_{\beta_2}(x))$ is star-shaped, i.e., $\frac{G_{\beta_1}^{-1}(G_{\beta_2}(x))}{x}$ is an increasing function of $x \in$

* Corresponding author. Tel.: +91 9336340459.
E-mail addresses: neeraj@iitk.ac.in (N. Misra), arshadm@iitk.ac.in, arshad.iitk@gmail.com (M. Arshad).

0167-7152/$ – see front matter
http://dx.doi.org/10.1016/j.spl.2013.11.013

$(0, \infty)$ (see Dharmadhikari and Joag-dev (1988), Chapter 9). Using convex ordering property of gamma distributions, Barlow and Gupta (1969) established that $I_1(\alpha)$ is increasing in $\alpha \in (0, \infty)$. Alam (1970) studied the monotonicity of the integral

$$I_2(\alpha) = \int_0^\infty (1 - G_\alpha(dt))^{k-1}\, dG_\alpha(t), \quad \alpha > 0, \tag{1.3}$$

where $d \in (0, \infty)$ and $k \in \{2, 3, \ldots\}$ are fixed constants. They proved that for $0 < d < 1$ $(d > 1)$, $I_2(\alpha)$ is an increasing (decreasing) function of $\alpha \in (0, \infty)$. Panchapakesan (1978) provided sufficient conditions for the monotonicity of a general integral of the form $I_1(\alpha)$ under the assumption that $G_\alpha(\cdot)$ belongs to some general family of distribution functions and α is integer-valued. He also showed that these sufficient conditions are satisfied by the gamma family of distributions. Recently, for $k = 2$ and integer-valued α, McDonald and Panchapakesan (2006) provided an alternate proof for the monotonicity of integral $I_1(\alpha)$. In this paper, we shall unify the results of Barlow and Gupta (1969) and Alam (1970) on monotonicity of integrals $I_1(\alpha)$ and $I_2(\alpha)$ in two directions by studying the monotonicity of integrals

$$\Psi_1(\alpha) = \int_0^\infty \int_0^{d^* s} \prod_{j=2}^{k-1} \left(G_\alpha\left(\frac{s}{d_{2,j}}\right) - G_\alpha\left(d_{1,j}t\right)\right) dG_\alpha(t)dG_\alpha(s), \quad \alpha > 0, \tag{1.4}$$

and

$$\Psi_2(\alpha) = \int_0^\infty \left(G_\alpha\left(\frac{t}{d_2}\right) - G_\alpha\left(d_1 t\right)\right)^{k-1} dG_\alpha(t), \quad \alpha > 0, \tag{1.5}$$

where $d_{1,j} \in (0, 1)$, $d_{2,j} \in (0, 1)$, $j = 2, \ldots, k-1$, $d^* \in \left[1, \min\left(\frac{1}{d_{1,2}d_{2,2}}, \ldots, \frac{1}{d_{1,k-1}d_{2,k-1}}\right)\right]$, $d_1 \in (0, 1)$, $d_2 \in (0, 1)$, and $k \in \{2, 3, \ldots\}$ are fixed constants.

We will see in Section 2 that the integrals $I_1(\alpha)$ and $I_2(\alpha)$, defined in (1.2) and (1.3), respectively, are particular cases of each of the integrals $\Psi_1(\alpha)$ and $\Psi_2(\alpha)$, defined in (1.4) and (1.5) respectively. In Section 2, we will show that the integrals $\Psi_1(\alpha)$ and $\Psi_2(\alpha)$ are increasing functions of $\alpha \in (0, \infty)$, thereby generalizing the results of Barlow and Gupta (1969), Alam (1970), Panchapakesan (1978) and McDonald and Panchapakesan (2006). In Section 3, we will provide applications of these results to multiple comparison procedures and ranking and selection problems.

2. Main results

The following lemma will be useful in deriving the results of this section.

Lemma 2.1. *Let $G_\alpha(\cdot)$ be defined by (1.1). Then, for $0 < \alpha_1 < \alpha_2 < \infty$, $c_1 \in (0, 1)$, $c_2 \in (0, 1)$, and for all $x \in (0, \infty)$,*

(i) $\frac{G_{\alpha_1}^{-1}(G_{\alpha_2}(c_1 x))}{c_1 x} \le \frac{G_{\alpha_1}^{-1}(G_{\alpha_2}(x))}{x}$, *i.e.*, $G_{\alpha_1}^{-1}(G_{\alpha_2}(c_1 x)) \le c_1\, G_{\alpha_1}^{-1}(G_{\alpha_2}(x))$;

(ii) $\frac{G_{\alpha_1}^{-1}(G_{\alpha_2}(x))}{x} \le \frac{G_{\alpha_1}^{-1}(G_{\alpha_2}(\frac{x}{c_2}))}{\frac{x}{c_2}}$, *i.e.*, $\frac{1}{c_2}\, G_{\alpha_1}^{-1}(G_{\alpha_2}(x)) \le G_{\alpha_1}^{-1}\left(G_{\alpha_2}\left(\frac{x}{c_2}\right)\right)$.

The above lemma is an immediate consequence of star-shaped ordering of gamma distribution (i.e., for $0 < \alpha_1 < \alpha_2 < \infty$, $\frac{G_{\alpha_1}^{-1}(G_{\alpha_2}(x))}{x}$ is increasing in $x \in (0, \infty)$).

We will first prove that the integral $\Psi_1(\alpha)$, defined in (1.4), is increasing in $\alpha \in (0, \infty)$, i.e., $\Psi_1(\alpha_2) \ge \Psi_1(\alpha_1)$, whenever $\alpha_2 > \alpha_1 > 0$. Assume that $0 < \alpha_1 < \alpha_2$. Using (1.4), we have

$$\begin{aligned}\Psi_1(\alpha_1) &= \int_0^\infty \int_0^{d^* s} \prod_{j=2}^{k-1} \left(G_{\alpha_1}\left(\frac{s}{d_{2,j}}\right) - G_{\alpha_1}\left(d_{1,j}t\right)\right) dG_{\alpha_1}(t)dG_{\alpha_1}(s) \\ &= \int_0^\infty \int_0^\infty \prod_{j=2}^{k-1} \left(G_{\alpha_1}\left(\frac{G_{\alpha_1}^{-1}(G_{\alpha_2}(y))}{d_{2,j}}\right) - G_{\alpha_1}\left(d_{1,j}G_{\alpha_1}^{-1}(G_{\alpha_2}(x))\right)\right) \\ &\quad \times I\left(G_{\alpha_1}^{-1}(G_{\alpha_2}(x)) \le d^* G_{\alpha_1}^{-1}(G_{\alpha_2}(y))\right) dG_{\alpha_2}(x)dG_{\alpha_2}(y),\end{aligned} \tag{2.1}$$

where $d_{1,j} \in (0, 1)$, $d_{2,j} \in (0, 1)$, $j = 2, \ldots, k-1$, $d^* \in \left[1, \min\left(\frac{1}{d_{1,2}d_{2,2}}, \ldots, \frac{1}{d_{1,k-1}d_{2,k-1}}\right)\right]$, and $I(A) = 1$, if A holds, and $= 0$, otherwise. Since $d_{1,j} \in (0, 1)$, $d_{2,j} \in (0, 1)$, $j = 2, \ldots, k-1$, and $d^* \ge 1$, using Lemma 2.1 in (2.1), we get

$$\begin{aligned}\Psi_1(\alpha_1) &\le \int_0^\infty \int_0^\infty \prod_{j=2}^{k-1} \left(G_{\alpha_1}\left(G_{\alpha_1}^{-1}\left(G_{\alpha_2}\left(\frac{y}{d_{2,j}}\right)\right)\right) - G_{\alpha_1}\left(G_{\alpha_1}^{-1}(G_{\alpha_2}(d_{1,j}x))\right)\right) \\ &\quad \times I\left(G_{\alpha_1}^{-1}(G_{\alpha_2}(x)) \le G_{\alpha_1}^{-1}(G_{\alpha_2}(d^* y))\right) dG_{\alpha_2}(x)dG_{\alpha_2}(y) \\ &= \int_0^\infty \int_0^{d^* y} \prod_{j=2}^{k-1} \left(G_{\alpha_2}\left(\frac{y}{d_{2,j}}\right) - G_{\alpha_2}\left(d_{1,j}x\right)\right) dG_{\alpha_2}(x)dG_{\alpha_2}(y) \\ &= \Psi_1(\alpha_2).\end{aligned}$$