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Slow recurrent regimes for a class of one-dimensional stochastic growth models

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Received 13 May 2016; received in revised form 7 October 2017; accepted 21 October 2017
Available online xxxx

Abstract

We classify the possible behaviors of a class of one-dimensional stochastic recurrent growth models. In our main result, we obtain nearly optimal bounds for the tail of hitting times of some compact sets. If the process is an aperiodic irreducible Markov chain, we determine whether it is null recurrent or positive recurrent and in the latter case, we obtain a subgeometric convergence of its transition kernel to its invariant measure. We apply our results in particular to state-dependent Galton–Watson processes and we give precise estimates of the tail of the extinction time.

MSC: 60J10; 60J80

Keywords: Markov chains; Hitting-times; Recurrence classification; Lyapunov function; Stochastic difference equation

1. Introduction and main result

1.1. Introduction

We consider a stochastic growth model $(X_n)_{n\in\mathbb{N}}$, taking values in $\mathcal{X}$, an unbounded subset of $\mathbb{R}_+$, and satisfying a stochastic difference equation of the form

$$X_{n+1} = X_n + g\left(X_n\right) + \xi_n, \tag{1}$$

where g is a given function and $(\xi_n)_{n\in\mathbb{N}}$ is a sequence of random variables such that almost surely,

$$\mathbb{E}\left(\xi_n \middle| \mathcal{F}_n\right) = 0,$$

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https://doi.org/10.1016/j.spa.2017.10.005
0304-4149/

$$\mathbb{E}\left(\xi_n^2 \middle| \mathcal{F}_n\right) = \sigma^2(X_n) < \infty,$$

for some positive function $\sigma^2(x)$. The filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$ is such that $(X_n)_{n\in\mathbb{N}}$ is $\mathcal{F}_n$ measurable for all $n \in \mathbb{N}$.

Provided that the following limit exists

$$\theta = \lim_{x\to\infty} \frac{2xg(x)}{\sigma^2(x)},$$

and belongs to $(-\infty, 1)$, Kersting proved in [10] that $\mathbb{P}(\{X_n \underset{n\to\infty}{\longrightarrow} \infty\}) = 0$ and said that $(X_n)_{n\in\mathbb{N}}$ is recurrent, adopting the terminology from Markov chain theory, whereas if $\theta \in (1, \infty)$ then $\mathbb{P}(\{X_n \underset{n\to\infty}{\longrightarrow} \infty\}) > 0$. A similar criterion for the multidimensional case was recently given in [1].

The aim of this article is to determine how quickly the process $(X_n)_{n\in\mathbb{N}}$, started from $x > A$, goes into the interval $[0, A]$, where $A > 0$ is arbitrary. If $(X_n)_{n\in\mathbb{N}}$ is an aperiodic irreducible Markov chain, we deduce therefrom a criterion of positive recurrence and how fast $(X_n)_{n\in\mathbb{N}}$ converges to its invariant measure. Moreover, if we have in mind population models, where a natural assumption is the dichotomy property, *i.e.*,

$$\mathbb{P}\left(\left\{X_n \underset{n\to\infty}{\longrightarrow} \infty\right\}\right) + \mathbb{P}\left(\{\exists n \text{ such that } X_n = 0\}\right) = 1,$$

we obtain precise estimates of the tail of the extinction time.

The first key ingredient of this article is to consider power functions as Lyapunov functions for growth models. Kersting [10] proved recurrence and transience of growth models by using the logarithm as a Lyapunov function. However, we cannot get more information on the behavior of $(X_n)_{n\in\mathbb{N}}$ with this function. Considering power functions yields an inequality of the form

$$\mathbb{E}\left(X_{n+1}^\alpha \middle| \mathcal{F}_n\right) - X_n^\alpha \le -C X_n^{\alpha-1} g(X_n) + b\mathbb{1}_{\{X_n \le A\}},$$

for all $n \in \mathbb{N}$, where $\alpha \in (0, 1)$, A, C and b some positive constants. From this equation, we deduce that

$$\mathbb{E}\left(f(Y_{n+1}) \middle| \mathcal{F}_n\right) - f(Y_n) \le -Cf'(Y_n) + b\mathbb{1}_{\{Y_n \le A\}}, \tag{2}$$

where Y_n is a transform of X_n, f is an increasing differentiable function, and A, C and b are some positive constants. Inequality (2) enables us to give all possible behaviors of our class of recurrent growth models. In a series of papers [3–5], Aspandiiarov et al. proved upper and lower bounds for the tail of hitting-time into compact sets, for processes verifying some conditions, improving previous results of Lamperti [13]. The second key ingredient, is to apply these results on a transform $Y_n = G(X_n)$ of our process to get an upper bound of hitting-time into compact sets. If $(X_n)_{n\in\mathbb{N}}$ is an aperiodic irreducible Markov chain, we give a criterion for null recurrence or positive recurrence. Moreover, if $(X_n)_{n\in\mathbb{N}}$ is positive recurrent, we obtain from [4] in the countable state space, from [7] in a general state space, subgeometric rate of convergence to its invariant probability measure. Thus, we give a complete classification of behaviors of stochastic recurrent growth processes of the form (1). By applying our results, we deduce nearly optimal upper and lower bounds of the tail of the extinction time of state-dependent Galton–Watson processes that seem to have never been studied before, to the best of our knowledge. We also recover a weaker version of results of Zubkov [16] on the return time to zero of critical Galton–Watson process with immigration, but without using probability generating functions.

The article is organized as follows. In the next subsection, our main results Theorems 1.1 and 1.2 are stated. Then, in Section 2 we state and prove a series of lemmas needed for the proof