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Pointwise estimates for first passage times of perpetuity sequences

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Abstract

We consider first passage times $\tau_u = \inf\{n : Y_n > u\}$ for the perpetuity sequence

$$Y_n = B_1 + A_1 B_2 + \cdots + (A_1 \ldots A_{n-1}) B_n,$$

where (A_n, B_n) are i.i.d. random variables with values in $\mathbb{R}^+ \times \mathbb{R}$. Recently, a number of limit theorems related to τ_u were proved including the law of large numbers, the central limit theorem and large deviations theorems (see Buraczewski et al., in press). We obtain a precise asymptotics of the sequence $\mathbb{P}[\tau_u = \log u/\rho]$, $\rho > 0$, $u \to \infty$ which considerably improves the previous results of Buraczewski et al. (in press). There, probabilities $\mathbb{P}[\tau_u \in I_u]$ were identified, for some large intervals I_u around k_u, with lengths growing at least as $\log \log u$. Remarkable analogies and differences to random walks (Buraczewski and Maślanka, in press; Lalley, 1984) are discussed.

MSC: primary 60H25; secondary 60F10; 60J10

Keywords: Random recurrence equations; Stochastic fixed point equations; First passage times; Large deviations; Asymptotic behavior; Ruin probabilities

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1. Introduction

1.1. The perpetuity sequence and first passage times

Let (A_n, B_n) be i.i.d. (independent identically distributed) random variables with values in $\mathbb{R}^+ \times \mathbb{R}$. We consider the perpetuity sequence

$$Y_n = B_1 + A_1 B_2 + \cdots + (A_1 \cdots A_{n-1}) B_n, \quad n = 1, 2, \ldots.$$

If $\mathbb{E}[\log A] < 0$ and $\mathbb{E}[\log^+ |B|] < \infty$, Y_n converges a.s. to the random variable

$$Y = \sum_{n=0}^{\infty} A_1 \ldots A_n B_{n+1},$$

that is the unique solution to the random difference equation

$$Y \stackrel{d}{=} AY + B, \qquad Y \text{ independent of } (A, B).$$

The perpetuity process $\{Y_n\}$ is frequently present in both applied and theoretical problems. On one hand, the perpetuity sequence plays an important role in analyzing the ARCH and GARCH financial time series models, see Engle [9] and Mikosch [18]. On the other, it is connected to the random walk in random environment [15], the weighted branching process and the branching random walk, see Guivarc'h [12], Liu [17] and Buraczewski [1]. We refer the reader to recent monographs [4,13] for an overview on the subject.

The main objective of this paper is to study the asymptotic behavior of the first passage time

$$\tau_u := \inf\{n : Y_n > u\} \quad \text{as } u \to \infty.$$

This is a basic question motivated partly by similar problems considered for random walks $\log \Pi_n = \log(A_1 \ldots A_n)$, see the work of Lalley [16]. Some partial results in this direction were proved by Nyrhinen [19,20]. An essential progress has been recently achieved in [2] and the aim of the present paper is to pursue the investigation further.

The tail of Y was analyzed under the Cramér condition

$$\mathbb{E}\big[A^{\alpha_0}\big] = 1 \text{ for some } \alpha_0 > 0. \tag{1.1}$$

Then Kesten [14] and Goldie [11] proved that[1]

$$\mathbb{P}[Y > u] \sim c_+ u^{-\alpha_0} \quad \text{as } u \to \infty, \tag{1.2}$$

which entails

$$\mathbb{P}[\tau_u < \infty] \sim c_0 u^{-\alpha_0} \quad \text{as } u \to \infty. \tag{1.3}$$

In [2] the authors proved the law of large numbers ([2], Lemma 2.1)

$$\frac{\tau_u}{\log u} \bigg| \tau_u < \infty \xrightarrow{\mathbb{P}} \frac{1}{\rho_0}$$

and the central limit theorem ([2], Theorem 2.2)

$$\frac{\tau_u - \log u/\rho_0}{\sigma_0 \rho_0^{-3/2} \sqrt{\log u}} \bigg| \tau_u < \infty \xrightarrow{d} N(0, 1), \tag{1.4}$$

[1] We write $f(u) \sim g(u)$ for two functions f and g, if $f(u)/g(u) \to 1$ as $u \to \infty$.

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