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Markov processes with darning and their approximations

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Abstract

In this paper, we study darning of general symmetric Markov processes by shorting some parts of the state space into singletons. A natural way to construct such processes is via Dirichlet forms restricted to the function spaces whose members take constant values on these collapsing parts. They include as a special case Brownian motion with darning, which has been studied in details in Chen (2012), Chen and Fukushima (2012) and Chen et al. (2016). When the initial processes have discontinuous sample paths, the processes constructed in this paper are the genuine extensions of those studied in Chen and Fukushima (2012). We further show that, up to a time change, these Markov processes with darning can be approximated in the sense of finite-dimensional distributions by introducing additional jumps with large intensity among these compact sets to be collapsed into singletons. For diffusion processes, it is also possible to get, up to a time change, diffusions with darning by increasing the conductance on these compact sets to infinity. To accomplish these, we give a version of the semigroup characterization of Mosco convergence to closed symmetric forms whose domain of definition may not be dense in the L^2-space. The latter is of independent interest and potentially useful to study convergence of Markov processes having different state spaces. Indeed, we show in Section 5 of this paper that Brownian motion in a plane with a very thin flag pole can be approximated by Brownian motion in the plane with a vertical cylinder whose horizontal motion on the cylinder is a circular Brownian motion moving at fast speed.

MSC: primary 60J25; 31C25; secondary 60F99

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1. Introduction

K. Ito [18] introduced the notion of Poisson point process of excursions around one point a in the state space of a standard Markov process X. He was motivated by giving a systematic construction of Markovian extensions of the absorbing diffusion process X^0 on the half line $(0, \infty)$ subject to Feller's general boundary conditions [19]. Ito had constructed the most general jump-in process from the exit boundary 0 by using Poisson point process of excursions. Recent study [14,6,2] reveals that Ito's program works equally well in the study of Markov processes transformed by collapsing certain compact subsets of the state space into singletons. These processes are called Markov processes with darning in [2]. (When the underlying process is a Markov chain on a discrete state space, such a procedure of collapsing subsets of state space is also called shorting in some literature.) However, in order to use excursion theory, it is assumed in [14,6,2] that the original Markov process enters these compact subsets in a continuous way. This condition is automatically satisfied for diffusion processes but not for general symmetric Markov processes that may have discontinuous trajectories.

The purpose of this paper is two-folds. First, we extend the notion and construction of Markov processes with darning to any symmetric Markov process, without assuming that the processes enter the compact subsets to be collapsed in a continuous way. In this generality, we can no longer use Poisson point process of excursions for the construction. We will use instead a Dirichlet form approach, which turns out to be quite effective. The second goal is to investigate approximation schemes for general Markov processes with darning by more concrete processes, which can be used for simulation. For this, we present a version of Mosco convergence of closed symmetric forms whose domain may not be dense in the underlying L^2-space. This is because due to the collapsing of the compact holes, the domain of the Dirichlet form for the Markov process with darning is not dense in the L^2-space on the original state space. We now describe the content of this paper in some details. For basic definitions and properties of symmetric Dirichlet forms, we refer the reader to [2,13].

Let E be a locally compact separable metric space and m a Radon measure on E with full support. Suppose $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on $L^2(E; m)$ in the sense that $C_c(E) \cap \mathcal{F}$ is dense both in $C_c(E)$ with respect to the uniform norm in $\mathcal{F}$ and with respect to the Hilbert norm $\sqrt{\mathcal{E}_1(u, u)} := \sqrt{\mathcal{E}(u, u) + (u, u)_{L^2(E;m)}}$. Here and in the sequel, we use := as a way of definition and $C_c(E)$ is the space of continuous functions on E with compact support. Every f in $\mathcal{F}$ admits an $\mathcal{E}$-quasi-continuous m-version, which is unique up to an $\mathcal{E}$-polar set. We always take such a quasi-continuous version for functions in $\mathcal{F}$. There is an m-symmetric Hunt process X on E associated with $(\mathcal{E}, \mathcal{F})$, which is unique up to an $\mathcal{E}$-polar set. It is known that for any regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(E; m)$ it admits the following unique Beurling–Deny decomposition (see [2,13]):

$$\mathcal{E}(u, u) = \mathcal{E}^c(u, u) + \frac{1}{2}\int_{E\times E}(u(x) - u(y))^2 J(dx, dy) + \int_E u(x)^2 \kappa(dx), \qquad u \in \mathcal{F},$$

where $\mathcal{E}^c$ is a symmetric non-negative definite bilinear form on $\mathcal{F}$ that satisfies strong local property, where $J(dx, dy)$ is a σ-finite measure on $E \times E \setminus$ diagonal, and κ is a σ-finite smooth measure on E. The measures $J(dx, dy)$ and $\kappa(dx)$ are called the jumping measure and killing measure of the process X (or equivalently, of the Dirichlet form $(\mathcal{E}, \mathcal{F})$). Indeed, if we use $(N(x, dy), H_t)$ to denote a Lévy system of X, where $N(x, dy)$ is a kernel on $E_\partial := E \cup \{\partial\}$ and $t \mapsto H_t$ is a positive continuous additive functional (PCAF) of X, then

$$J(dx, dy) = N(x, dy)\mu_H(dx) \qquad \text{and} \qquad \kappa(dx) = N(x, \{\partial\})\mu_H(dx).$$