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# Discretizing Malliavin calculus

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**Abstract**

Suppose $B$ is a Brownian motion and $B^n$ is an approximating sequence of rescaled random walks on the same probability space converging to $B$ pointwise in probability. We provide necessary and sufficient conditions for weak and strong $L^2$-convergence of a discretized Malliavin derivative, a discrete Skorokhod integral, and discrete analogues of the Clark–Ocone derivative to their continuous counterparts. Moreover, given a sequence $(X^n)$ of random variables which admit a chaos decomposition in terms of discrete multiple Wiener integrals with respect to $B^n$, we derive necessary and sufficient conditions for strong $L^2$-convergence to a $\sigma(B)$-measurable random variable $X$ via convergence of the discrete chaos coefficients of $X^n$ to the continuous chaos coefficients.


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## 1. Introduction

Let $B = (B_t)_{t\geq 0}$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$, where the $\sigma$-field $\mathcal{F}$ is generated by the Brownian motion and completed by null sets. Suppose $\xi$ is a square-integrable random variable with zero expectation and variance one. As a discrete counterpart of

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$B$ we consider, for every $n \in \mathbb{N} = \{1, 2, \dots\}$, a random walk approximation

$$B_t^n := \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} \xi_i^n \ , \ t \geq 0,$$

where $(\xi_i^n)_{i\in\mathbb{N}}$ is a sequence of independent random variables which have the same distribution as $\xi$. We assume that the approximating sequence $B^n$ converges to $B$ pointwise in probability, i.e.

$$\forall\, t \geq 0 : \quad \lim_{n\to\infty} B_t^n = B_t \ \text{ in probability.} \tag{1}$$

The aim of the paper is to provide $L^2$-approximation results for some basic operators of Malliavin calculus with respect to the Brownian motion $B$ such as the chaos decomposition, the Malliavin derivative, and the Skorokhod integral by appropriate sequences of approximating operators based on the discrete time noise $(\xi_i^n)_{i\in\mathbb{N}}$. It turns out that in all our approximation results, the limits do not depend on the distribution of the discrete time noise, hence our results can be regarded as some kind of invariance principle for Malliavin calculus.

We briefly discuss our main convergence results in a slightly informal way:

(1) *Chaos decomposition*: The heuristic idea behind the chaos decomposition in terms of multiple Wiener integrals is to project a random variable $X \in L^2(\Omega, \mathcal{F}, P)$ on products of the white noise $\dot{B}_{t_1} \cdots \dot{B}_{t_k}$. This idea can be made rigorous with respect to the discrete noise $(\xi_i^n)_{i\in\mathbb{N}}$ by considering the discrete time functions

$$f_{k,X}^n(i_1, \dots, i_k) = \frac{n^{k/2}}{k!} \mathbb{E}\left[X \prod_{j=1}^{k} \xi_{i_j}^n\right]$$

for pairwise distinct $(i_1, \dots, i_k) \in \mathbb{N}^k$. Our results show that, after a natural embedding as step functions into continuous time, the sequence $(f_{k,X}^n)_{n\in\mathbb{N}}$ converges strongly in $L^2([0,\infty)^k)$ to the $k$th chaos coefficient of $X$, for every $k \in \mathbb{N}$ (Example 34). This is a simple consequence of a general Wiener chaos limit theorem (Theorem 28), which provides equivalent conditions for the strong $L^2(\Omega, \mathcal{F}, P)$-convergence of a sequence of random variables $(X^n)_{n\in\mathbb{N}}$ (with each $X^n$ admitting a chaos decomposition via multiple Wiener integrals with respect to the discrete time noise $(\xi_i^n)_{i\in\mathbb{N}}$) in terms of the chaos coefficient functions. As a corollary, this Wiener chaos limit theorem lifts a classical result in [33] on convergence in distribution of discrete multiple Wiener integrals to strong $L^2(\Omega, \mathcal{F}, P)$-convergence and adds the converse implication (in our setting, i.e. when the limiting multiple Wiener integral is driven by a Brownian motion).

(2) *Malliavin derivative:* With our weak moment assumptions on the discrete time noise, we cannot define a discrete Malliavin derivative in terms of a polynomial chaos as in the survey paper [12] and the references therein. Instead we think of the discretized Malliavin derivative at time $j \in \mathbb{N}$ with respect to the noise $(\xi_i^n)_{i\in\mathbb{N}}$ as

$$D_j^n X = \sqrt{n}\mathbb{E}[\xi_j^n X | (\xi_i^n)_{i\in\mathbb{N}\setminus\{j\}}],$$

which is the gradient of the best approximation in $L^2(\Omega, \mathcal{F}, P)$ of $X$ as a linear function in $\xi_j^n$ with $\sigma(\xi_i^n,\ i \in \mathbb{N} \setminus \{j\})$-measurable coefficients. In the case of binary noise, this definition coincides with the standard notion of the Malliavin derivative on the Bernoulli space, see, e.g., [29]. Theorem 12 below implies that, if $(X^n)$ converges weakly in $L^2(\Omega, \mathcal{F}, P)$ to $X$ and the sequence of discretized Malliavin derivatives $(D_{\lceil n\cdot\rceil}^n X^n)_{n\in\mathbb{N}}$

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