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Transportation distances and noise sensitivity of multiplicative Lévy SDE with applications

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Abstract

This article assesses the distance between the laws of stochastic differential equations with multiplicative Lévy noise on path space in terms of their characteristics. The notion of transportation distance on the set of Lévy kernels introduced by Kosenkova and Kulik yields a natural and statistically tractable upper bound on the noise sensitivity. This extends recent results for the additive case in terms of coupling distances to the multiplicative case. The strength of this notion is shown in a statistical implementation for simulations and the example of a benchmark time series in paleoclimate.

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1. Introduction

Many dynamical phenomena are subject to random forcing, often described by stochastic differential equations of the following type

$$dX(t) = -\nabla U(X(t))dt + d\xi(t), \qquad X_0 = x_0, \tag{1.1}$$

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where $-\nabla U$ encodes the deterministic dynamics given by a potential gradient and ξ a noise signal. In general, for instance when ξ exhibits discontinuities, it is not straightforward to describe the law of the solution X on path space in terms of the parameters which determine the distribution of ξ. Our approach allows to quantify the distance of such laws in terms of accessible quantities, both analytically and statistically.

The case where ξ is given as a discontinuous Lévy process was studied in a previous publication [9]. The authors introduced the notion of a coupling distance between Lévy measures in order to quantify the Wasserstein distance on path space. The coupling distances have been found to be sufficiently strong (in a topological sense) to quantify the convergence in functional limit theorems, yet being weak enough in order to be numerically and statistically tractable. See for instance the calibration problem of a climate time series in [10]. In many situations, however, the noise process ξ exhibits state dependence, for instance multiplicative noise. This generalization lifts Lévy diffusions to Lévy-type diffusions. To treat this class of noise processes the authors introduced in [22] the notion of transportation distance extending the coupling distances with the help of a common reference Lévy measure. The present article establishes analogous bounds on the distance between the laws of Lévy-type diffusions on path space in terms of transportation distances.

We stress that our procedure is suitable for a large variety of phenomena modeled with jump diffusions, such as in finance, e.g. [25], or neurosciences [4,8,28]. The particular application we have in mind in this article is the refinement of the analysis of the noise structure behind the paleoclimate temperature evolution studied in [10]. The climate data apparently fluctuate around two distinct metastable states with rapid transitions (see Fig. 2). Such phenomena are observed in stochastic energy balance models [1–3,6,14,16,18–21,23]. There is a list of publications associating this time series to an underlying jump diffusion [11], see for instance [7,10,12,15]. Using various techniques these articles aim to determine the (polynomial) jump behavior of (1.1) for different classes of heavy-tailed Lévy processes ξ. The models so far require the spacial homogeneity of the noise characteristics. The present article lifts this restriction. We may now investigate the statical behavior in the different spatial regimes, prescribed by the metastable states, and solve the corresponding model selection problem on the generic class of heavy-tailed jump diffusions. Our results of the implementation of this program applied to the mentioned climate times series are consistent with the findings in [10].

2. Preliminaries

Transportation functions and transportation distance: Consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geqslant 0})$ satisfying the usual conditions in the sense of Protter [24] carrying a scalar Brownian motion $(B_t)_{t\in[0,T]}$ and an independent Cauchy–Poisson random measure ν_0 on $[0,T]\times\mathbb{R}$ with intensity measure $dt\otimes\Pi_0$ given by

$$\Pi_0(dv) = \mathbf{1}_{\{v\in\mathbb{R}\setminus\{0\}\}}\frac{dv}{v^2}.$$

We define a Lévy measure to be a σ-finite Borel measure on $\mathbb{R}\setminus\{0\}$ satisfying

$$\int_{\mathbb{R}} (|v|^2 \wedge 1)\Pi(dv) < \infty. \tag{2.1}$$

In contrast to the standard definition we do not exclude point-mass in 0 which may be taken to be infinity. Nevertheless we will identify all such measures that coincide on the Borel σ-algebra