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Homogenization of dissipative, noisy, Hamiltonian dynamics

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Abstract

We study the dynamics of a class of Hamiltonian systems with dissipation, coupled to noise, in a singular (small mass) limit. We derive the homogenized equation for the position degrees of freedom in the limit, including the presence of a *noise-induced drift* term. We prove convergence to the solution of the homogenized equation in probability and, under stronger assumptions, in an L^p-norm. Applications cover the overdamped limit of particle motion in a time-dependent electromagnetic field, on a manifold with time-dependent metric, and the dynamics of nuclear matter.

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1. Introduction

In the simplest case, the motion of a diffusing particle of non-zero mass, m, is governed by a stochastic differential equation (SDE) of the form

$$dq_t = v_t dt, \quad m dv_t = -\gamma v_t dt + \sigma dW_t, \tag{1.1}$$

where γ and σ are the dissipation (or drag) and diffusion coefficients respectively and W_t is a Wiener process. The study of diffusive systems in the limit $m \to 0$ was initiated by

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Smoluchowski in [27] and continued by Kramers in [18]. The field has grown to explore a large array of models and phenomena, including coupled fluid–particle systems [22], relativistic diffusion [2,8], and a variety of processes and convergence modes on manifolds [1,5,6,9,14,19,24,25]. History of the subject and a review of the early literature can be found in [20]. Such problems can be classified under the broad umbrella of homogenization, for which [23] is an excellent reference.

Recently, there has been increased interest in the phenomenon of *noise-induced drift*, which arises when the drag and noise coefficients are state dependent. In such cases, the equation governing the process in the limit $m \to 0$ possesses an additional drift term that was not present in the original system. First derived in [11], this has been observed experimentally in [28] and derived rigorously for one dimensional systems [26], systems satisfying the fluctuation–dissipation relation [11], in Euclidean space of arbitrary dimension [12,13], and on compact Riemannian manifolds of arbitrary dimension [3]. Further references to work on the phenomenon of noise-induced drift are found in [13].

Statistical mechanics of fluctuating systems, as reviewed in [7,10], covers systems more general than those governed by the Hamiltonians with quadratic kinetic energy,

$$H(q,p)=\frac{\|p\|^2}{2m}+V(q), \tag{1.2}$$

but to this point, the study of noise-induced drift has been restricted to Hamiltonians quadratic in p. In this paper, we extend the theory to a large class of Hamiltonian systems generalizing Eq. (1.2). See Section 2 for examples of the type of systems that are covered. We prove that solutions to these more general Hamiltonian systems converge in probability and, under stronger assumptions, in an L^p-norm to solutions of a homogenized limiting equation with a noise-induced drift term, for which we derive an explicit formula. This is a far-reaching generalization of the previous results about the $m \to 0$ limit of the equations Eq. (1.1).

1.1. Dissipative Hamiltonian system with noise

Here, we review the basic equations and properties of dissipative, noisy Hamiltonian systems. See also [7]. Given a time-dependent Hamiltonian $H(t,x)$ which is C^1 jointly in $t \in \mathbb{R}$ and $x=(q,p)\in\mathbb{R}^n\times\mathbb{R}^n$, a positive-semi-definite continuous matrix-valued function $\Gamma(t,x)$, the matrix

$$\Pi=\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}, \tag{1.3}$$

and a continuous vector field $G(t,x)$, we first consider the following deterministic equation

$$\dot{x}_t=-\Gamma(t,x_t)\nabla H(t,x_t)+\Pi\nabla H(t,x_t)+G(t,x_t). \tag{1.4}$$

This equation describes the dynamics of a dissipative Hamiltonian system with drag matrix Γ and external forcing G. We will refer to q as the position degrees of freedom and to p as the momentum degrees of freedom.

The rate of change of the Hamiltonian along a solution is given by

$$\begin{aligned}&\frac{d}{dt}H(t,x_t) \qquad (1.5)\\ &=\partial_t H(t,x_t)-\nabla H(t,x_t)\cdot\Gamma(t,x_t)\nabla H(t,x_t)\\ &\quad+\nabla H(t,x_t)\cdot\Pi\nabla H(t,x_t)+\nabla H(t,x_t)\cdot G(t,x_t)\end{aligned}$$