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Fractional diffusion-type equations with exponential and logarithmic differential operators

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Abstract

We deal with some extensions of the space-fractional diffusion equation, which is satisfied by the density of a stable process (see Mainardi et al. (2001)): the first equation considered here is obtained by adding an exponential differential (or shift) operator expressed in terms of the Riesz–Feller derivative. We prove that this produces a random component in the time-argument of the corresponding stable process, which is represented by the so-called Poisson process with drift. Analogously, if we add, to the space-fractional diffusion equation, a logarithmic differential operator involving the Riesz-derivative, we obtain, as a solution, the transition semigroup of a stable process subordinated by an independent gamma subordinator with drift. Finally, we show that an extension of the space-fractional diffusion equation, containing both the fractional shift operator and the Feller integral, is satisfied by the transition density of the process obtained by time-changing the stable process with an independent linear birth process with drift.

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1. Introduction

The diffusion equation has been generalized in the fractional sense by many authors (e.g. [41,38,8,2]): in particular [25] and [26] consider the time-fractional Cauchy problems, while in [14] the order of both time and space derivatives is fractional. Later, in [22] and [23],

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the time and space fractional diffusion equation was studied and solved analytically, also in the asymmetric case. The probabilistic expression of the solution to the diffusion equation with time-derivative of fractional order ν is given in [31], in terms of iterated stable processes (in particular, for $\nu = 1/2^n$, $n \in \mathbb{N}$, the n-times iterated Brownian motion).

The term 'anomalous diffusion' usually indicates a diffusive process that does not follow the behavior of classical Gaussian diffusions (see e.g. [28,11]). In the real world, anomalous diffusions are observed, for example, in turbulent plasma transport, photon diffusion, cell migration and so on. When a fractional derivative of order $\nu \in (0, 2)$ replaces the time-derivative in a diffusion model, we get an anomalous diffusion, for $\nu \neq 1$, since its mean squared displacement follows the power law in time, Dt^ν, with the constant coefficient D. In particular, for $\nu \in (0, 1)$ it behaves as a subdiffusion, slower than the classical one, while, for $\nu \in (1, 2)$, we get a superdiffusion, which is faster (see [22]).

The stochastic time–space fractional heat-type equation has been treated in [30]. For applications to physical and financial problems, see also [18,29,36]. Recently anomalous diffusions are often modeled by the so-called continuous-time random walks (CTRWs), which are defined as random walks subordinated to a counting renewal process (see [27,37]).

We consider here extensions of the following space-fractional diffusion equation, i.e.

$$\partial_t u(x,t) = \mathcal{D}_x^{\alpha,\theta} u(x,t),$$

where $\mathcal{D}_x^{\alpha,\theta}$ is the Riesz–Feller derivative of order $\alpha \in (0, 2]$, defined below. In particular we introduce in the above equation additional terms represented by the so-called fractional exponential (or shift) operator $\mathcal{O}_{c,x}^{\alpha,\theta}$ or the fractional logarithmic operator $\mathcal{P}_{c,x}^{\alpha}$ (see Definitions 1 and 4 below). We are thus led to study the following equations, again for $\alpha \in (0, 2]$,

$$\partial_t u(x,t) = \left[a\mathcal{D}_x^{\alpha,\theta} + \lambda (I - \mathcal{O}_{-1,x}^{\alpha,\theta}) \right] u(x,t) \tag{1.1}$$

$$\partial_t u(x,t) = \left[a\mathcal{D}_x^{\alpha} + \mu \mathcal{P}_{1/\rho,x}^{\alpha} \right] u(x,t) \tag{1.2}$$

(under appropriate initial and boundary conditions), where we denote by $\mathcal{D}_x^\alpha = \mathcal{D}_x^{\alpha,0}$ the symmetric Riesz derivative. We prove that the solution to Eq. (1.1) coincides with the transition semigroup of the subordinated process defined as $\mathcal{S}_{\alpha,\theta}(at + N(t))$, $t \geq 0$, where $\mathcal{S}_{\alpha,\theta} := \mathcal{S}_{\alpha,\theta}(t), t \geq 0$, is an α-stable process and $N := N(t), t \geq 0$, is an independent Poisson subordinator, with parameter λ. In the second case, we prove instead that Eq. (1.2) is satisfied by the transition semigroup of another subordinated process defined as $\mathcal{S}_\alpha(at + \Gamma(t))$, where $\mathcal{S}_\alpha := \mathcal{S}_{\alpha,0}$ is a symmetric α-stable process and $\Gamma(t)$, $t \geq 0$, is an independent gamma subordinator, with scale parameter $\mu > 0$. However, in both cases, the processes obtained are, for any $\alpha \in (0, 2)$, pure jump models, while, only for $\alpha = 2$, they have a jump–diffusion behavior. In particular, we notice that $\mathcal{S}_\alpha(at + \Gamma(t))$ reduces, for $\alpha = 2$ and $a = 0$, to the well-known Variance Gamma (VG) process. Jump–diffusions and VG processes are applied in finance, in particular for asset pricing (see e.g. [9]). For a general $\alpha \in (0, 2)$, the process $\mathcal{S}_\alpha(at + \Gamma(t))$ can be considered as a generalization of both stable and geometric stable processes (see, for example, [21]), to which it reduces in special cases.

In the last section we prove that an extension of (1.1), obtained by adding a Feller fractional integral, is satisfied by the transition density of a stable process time-changed by an independent linear birth process with drift.

Therefore, in all these cases, the additional operator introduced in the fractional diffusion equation entails the appearance of a random element in the time argument of the corresponding

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