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On the local times of stationary processes with conditional local limit theorems

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Abstract

We investigate the connection between conditional local limit theorems and the local time of integer-valued stationary processes. We show that a conditional local limit theorem (at 0) implies the convergence of local times to Mittag-Leffler distributions, both in the weak topology of distributions and a.s. in the space of distributions.

1. Introduction and main results

Local time characterizes the amount of time a process spends at a given level. Let $X_1, X_2, \ldots$ be integer-valued random variables, $S_n = \sum_{i=1}^{n} X_i$. The local time of $\{S_n\}$ on level x at time n is defined to be $\ell(n, x) := \#\{i = 1, 2, \ldots, n : S_i = x\}$. Denote by ℓ_n the local time at 0 at time n for short. For simple random walks, the exact and the limit distributions of ℓ_n are well known. Chung and Hunt (1949) [12] studied the limit behavior of the sequence of $\{\ell_n\}$. Révész (1981) [23] proved an almost sure invariance principle by Skorokhod embedding. For more general random walks, Borodin (1984) [8] established the weak convergence of $\ell(x\sqrt{n}, [nt])/\sqrt{n}$ of a recurrent random walk to the Brownian local time. Aleškevičienė (1986) [6] gave the asymptotic distribution and moments of local times of an aperiodic recurrent random walk. Bromberg and

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Kosloff (2012) [10] proved weak invariance principle for the local times of partial sums of Markov Chains. Bromberg (2014) [9] extended it to Gibbs–Markov processes.

In this article we study the connection of local times and local limit theorems as stated below.

Definition 1.1. A centered integer-valued stationary process $\{X_n\}$ is said to have a conditional local limit theorem at 0, if there exist a constant $g(0) > 0$ and a sequence $\{B_n\}$ of positive real numbers, such that for all $x \in \mathbb{Z}$

$$\lim_{n\to\infty} B_n P(S_n = x|(X_{n+1}, X_{n+2}, \ldots) = \cdot) = g(0) \tag{1.1}$$

almost surely.

The full formulation of the corresponding form of a local limit theorem goes back to Stone and reads in the conditional form (see [4]) that

$$\lim_{n\to\infty} B_n P(S_n = k_n|(X_{n+1}, X_{n+2}, \ldots) = \cdot) = g(\kappa) \quad \text{as } \frac{k_n - A_n}{B_n} \to \kappa \quad P\text{-a.s.} \tag{1.2}$$

for all $\kappa \in \mathbb{R}$, where A_n is some centering constant. Condition (1.1) can be reformulated using the dual operator P_T of the isometry $Uf = f \circ T$ on $L^\infty(P)$ operating on $L^1(P)$, where we take the L^p-spaces of P restricted to the σ-field generated by all X_n and T the shift operation $T(X_1, X_2, \ldots) = (X_2, X_3, \ldots)$. This operator is called the transfer operator. The local limit theorem at 0 then reads

$$\lim_{n\to\infty} B_n P_T^n(\mathbb{1}_{\{S_n = x\}}) = g(0) \qquad \text{for all } x \in \mathbb{Z},\ P\text{-a.s.} \tag{1.3}$$

In this paper, we assume that $\{X_n\}$ has the conditional local limit theorem at 0 as formulated in 1.1.

Remarks. For the full formulation (1.2), if the convergence is uniform for almost all ω and $\{X_n\}$ is strongly mixing, it would imply that $\{X_n\}$ has a local limit theorem; then by [16], $\{X_n\}$ are in the domain of attraction of a stable law with index d:

$$\frac{S_n - A_n}{B_n} \xrightarrow{\mathcal{W}} Z_d.$$

The probability density function of Z_d is g as above and the cumulative distribution function is denoted by $G(x)$. Since $\int_\Omega \phi\, dP = 0$, where ϕ is the same as the one in Section 2, we can (and will) assume that $A_n = 0$. It is necessary [16] that $\{B_n\}$ is regularly varying of order $\beta = 1/d$.

Next we state the main results of this paper.

Theorem 1.2 (*Convergence of Local Times*)**.** *Suppose that the integer-valued stationary process $\{X_n := \phi \circ T^{n-1} : n \geq 1\}$ has a conditional local limit theorem at* 0 (1.1) *with regularly varying scaling constants $B_n = n^\beta L(n)$, where $\beta \in [\frac{1}{2}, 1)$ and L is a slowly varying function. Put $a_n := g(0)\sum_{k=1}^n \frac{1}{B_k} \to \infty$. Then $\frac{\ell_n}{a_n}$ converges to a random variable Y_α strongly in distribution, i.e.*

$$\int_\Omega g\left(\frac{\ell_n(\omega)}{a_n}\right) H(\omega) dP(\omega) \to E[g(Y_\alpha)], \tag{1.4}$$

for any bounded and continuous function g and any probability density function H on $(\Omega, \mathcal{F}, P)$, and Y_α has the normalized Mittag-Leffler distribution of order $\alpha = 1 - \beta$.

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