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A sharp bound on the expected number of upcrossings of an L_2-bounded Martingale

David Gilat[a], Isaac Meilijson[a,*], Laura Sacerdote[b]

[a] School of Mathematical Sciences, R. and B. Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv 6997801, Israel

[b] Department of Mathematics G. Peano, University of Torino, Via Carlo Alberto 10, 10123 Torino, Italy

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Abstract

For a martingale M starting at x with final variance σ^2, and an interval (a, b), let $\Delta = \frac{b-a}{\sigma}$ be the normalized length of the interval and let $\delta = \frac{|x-a|}{\sigma}$ be the normalized distance from the initial point to the lower endpoint of the interval. The expected number of upcrossings of (a, b) by M is at most $\frac{\sqrt{1+\delta^2}-\delta}{2\Delta}$ if $\Delta^2 \leq 1 + \delta^2$ and at most $\frac{1}{1+(\Delta+\delta)^2}$ otherwise. Both bounds are sharp, attained by Standard Brownian Motion stopped at appropriate stopping times. Both bounds also attain the Doob upper bound on the expected number of upcrossings of (a, b) for submartingales with the corresponding final distribution. Each of these two bounds is at most $\frac{\sigma}{2(b-a)}$, with equality in the first bound for $\delta = 0$. The upper bound $\frac{\sigma}{2}$ on the length covered by M during upcrossings of an interval restricts the possible variability of a martingale in terms of its final variance. This is in the same spirit as the Dubins & Schwarz sharp upper bound σ on the expected maximum of M above x, the Dubins & Schwarz sharp upper bound $\sigma\sqrt{2}$ on the expected maximal distance of M from x, and the Dubins, Gilat & Meilijson sharp upper bound $\sigma\sqrt{3}$ on the expected diameter of M.

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* Corresponding author.
E-mail addresses: gilat@math.tau.ac.il (D. Gilat), isaco@math.tau.ac.il (I. Meilijson), laura.sacerdote@unito.it (L. Sacerdote).

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1. Introduction

Doob designed his upcrossings inequality [4] to prove the almost sure convergence of suitably bounded martingales, supermartingales and submartingales. Although various alternative proofs for the *martingale convergence theorem* were later presented (e.g. Chacon [1]), Doob's original proof based on the upcrossings inequality remains the standard textbook proof. As appropriately observed by Cinlar [2] and others, the proof of the upcrossings inequality can be reduced to the analysis of the situation for intervals of the form $(0, b)$ and non-negative submartingales. Whether reduced or not, the Doob upper bound for the expected number of upcrossings of an interval (a, b) by a submartingale with initial variable Y_0 and final variable Y is

The Doob upper bound: $\frac{1}{b-a}(E[(Y-a)^+] - E[(Y_0-a)^+])$.

It is relatively easy to construct submartingales that attain this bound: consider a *martingale* that may drop to a but not below a, followed by a deterministic increase from a to b. The concatenation of such blocks yields a submartingale $(Y_t - a)^+$ whose Doob decomposition [4] yields as the increasing process the number of upcrossings of (a, b) times the length $b - a$ of this interval, and attains the Doob bound.

The question addressed here is whether a martingale may attain the Doob bound, in the absence of an increasing process. Focus will be placed on studying upcrossing upper bounds for L_2-bounded martingales with deterministic initial value x and given variance $\sigma^2 > 0$ of its last term (or limit) Y. Let $l < r$, with $\frac{l+r}{2} = a$, be such that the dichotomous random variable with mean x supported by $\{l, r\}$ has variance σ^2.

The analysis will be divided into three cases, according in part to the relative position of the starting point x with respect to the target interval (a, b):

- Case I: $b < r$.
- Case II: $b \geq r$ and $x \leq a$.
- Case III: $b \geq r$ and $a < x < b$.

In case I, the expected number of upcrossings may exceed 1, while in cases II and III, it is at most 1. Since $x \in (l, r)$, if $x \geq b$, case I applies.

The maximal possible Doob upper bound under $Var(Y) = \sigma^2$ will be seen to be $\frac{\sqrt{\sigma^2+(x-a)^2}-|x-a|}{2(b-a)}$. In case I there is a martingale, Standard Brownian Motion (SBM) stopped at the first exit time from the interval (l, r), that attains this bound as equality. The optimality of this martingale for expected number of upcrossings is thus clear cut. In cases II and III the maximal expected number of upcrossings will be identified as well, but the argument is more involved. The case-II maximum is achieved by SBM stopped at the first exit time from an interval with b as upper endpoint. The case-III maximum is achieved by a two-stage stopping time in SBM, a first exit time from an interval with a as lower endpoint, followed only if a was reached in the first stage, by a first exit time from an interval with b as upper endpoint. Each of the two attains the Doob upper bound for the corresponding final distribution (supported by two or three atoms).

The worst-case role played by SBM can be appreciated from its nature as universal embedding environment for martingales—every martingale can be viewed as optional sampling of SBM (Monroe [8]). Hence, just as in Dubins & Schwarz [7] and Dubins, Gilat & Meilijson [5], it is enough to consider martingales of the very special form above—SBM stopped at appropriately defined stopping times. This is so because each of the maximum, the maximal absolute value, the diameter and the number of upcrossings of any interval, are a.s. bigger in the underlying SBM than in any L_2-bounded martingale embedded in it.

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