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Limit theorems for Hilbert space-valued linear processes under long range dependence

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Received 26 January 2017; received in revised form 23 May 2017; accepted 27 July 2017
Available online xxxx

Abstract

Let $(X_k)_{k\in\mathbb{Z}}$ be a linear process with values in a separable Hilbert space $\mathbb{H}$ given by $X_k = \sum_{j=0}^{\infty}(j+1)^{-N}\varepsilon_{k-j}$ for each $k \in \mathbb{Z}$, where $N : \mathbb{H} \to \mathbb{H}$ is a bounded, linear normal operator and $(\varepsilon_k)_{k\in\mathbb{Z}}$ is a sequence of independent, identically distributed $\mathbb{H}$-valued random variables with $E\varepsilon_0 = 0$ and $E\|\varepsilon_0\|^2 < \infty$. We investigate the central and the functional central limit theorem for $(X_k)_{k\in\mathbb{Z}}$ when the series of operator norms $\sum_{j=0}^{\infty}\|(j+1)^{-N}\|_{op}$ diverges. Furthermore, we show that the limit process in case of the functional central limit theorem generates an operator self-similar process.

Keywords: Linear processes; Long memory; Functional central limit theorem; Self-similarity; Hilbert space

1. Introduction

In this paper, we study long-range dependent linear processes with values in a separable Hilbert space $\mathbb{H}$. Given a sequence of bounded linear operators $u_j : \mathbb{H} \to \mathbb{H}$, $j \geq 0$ and a sequence of independent, identically distributed $\mathbb{H}$-valued random variables $(\varepsilon_k)_{k\in\mathbb{Z}}$ with $E\varepsilon_0 = 0$ and $E\|\varepsilon_0\|^2 < \infty$, we define the linear process

$$X_k = \sum_{j=0}^{\infty} u_j(\varepsilon_{k-j}), \quad k \in \mathbb{Z}. \tag{1}$$

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http://dx.doi.org/10.1016/j.spa.2017.07.015

We investigate the asymptotic distribution of the partial sums $S_n = \sum_{k=1}^{n} X_k$ and of the partial sums process $\zeta_n(t) = S_{\lfloor nt \rfloor} + \{nt\} X_{\lfloor nt \rfloor + 1}$ with $t \in [0, 1]$, where $\lfloor \cdot \rfloor$ denotes the floor function and $\{x\} = x - \lfloor x \rfloor$.

The behaviour of the linear process $(X_k)_{k\in\mathbb{Z}}$ crucially depends on the convergence respectively divergence of the series $\sum_{j=0}^{\infty} \|u_j\|_{op}$, where $\|\cdot\|_{op}$ denotes the operator norm. If $\sum_{j=0}^{\infty} \|u_j\|_{op} < \infty$, the process $(X_k)_{k\in\mathbb{Z}}$ is short range dependent. In this case, the central limit theorem holds with the usual normalizing sequence $n^{-\frac{1}{2}}$ and the normalized partial sums converge in distribution to an $\mathbb{H}$-valued Gaussian random element (see [15] and [14]). We are interested in the situation when the series diverges.

Račkauskas and Suquet [16] investigate a functional central limit theorem for $(X_k)_{k\in\mathbb{Z}}$ as in (1) with values in a Hilbert space $\mathbb{H}$ when $\sum_{j=0}^{\infty} \|u_j\|_{op}$ diverges with $u_0 = I$ and $u_j = j^{-T}$ for $j \geq 1$, where $T \in L(\mathbb{H})$ satisfies $\frac{1}{2} I < T < I$ and is self-adjoint. Additionally, they assume that the operator T commutes with the covariance operator of ε_0.

Characiejus and Račkauskas [4,5] consider $(X_k)_{k\in\mathbb{Z}}$ with values in the Hilbert space $L_2(\mu) = L_2(\mathbb{S}, \mathcal{S}, \mu)$ of square-integrable real-valued functions, where $(\mathbb{S}, \mathcal{S}, \mu)$ is a σ-finite measure space. They choose $u_j = (j + 1)^{-D}$ without requiring that the operator commutes with the covariance operator of ε_0. In their case D is a multiplication operator given by $Df = \{d(s) f(s) | s \in \mathbb{S}\}$ for each $f \in L_2(\mu)$ for a measurable function $d : \mathbb{S} \to \mathbb{R}$.

We combine both results, constructing a process with values in a complex Hilbert space $\mathbb{H}$ with inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\|\cdot\|$, choosing

$$u_j = (j + 1)^{-N} \tag{2}$$

for each $j \geq 0$, where $N \in L(\mathbb{H})$ is a normal operator, i.e. N commutes with its hermitian adjoint denoted by N^*, that is $NN^* = N^*N$.

To be more precise we give some details about operators. Let $A \in L(\mathbb{H})$, then it is called non-negative if $\langle Ax, x \rangle \geq 0$ for all $x \in \mathbb{H}$. For an additional operator $B \in L(\mathbb{H})$ the inequality $A \geq B$ means $A - B \geq 0$. We set $\exp(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!}$ and $a^A = \exp(A \log a)$ for $a > 0$. For further details about operators we refer to Comway [6] and Akhiezer and Glazman [1].

Our main results establish sufficient conditions for a central and a functional central limit theorem. More precisely we show convergence in distribution of $n^{-H} S_n$ in $\mathbb{H}$ and of $n^{-H} \zeta_n$ in the space $C([0, 1], \mathbb{H})$ to a Gaussian stochastic process with $H = \frac{3}{2} I - N$, where N is a normal operator and $C([0, 1], \mathbb{H})$ is the Banach space of continuous functions $x : [0, 1] \to \mathbb{H}$ endowed with the norm $\|x\| = \sup_{0\leq t\leq 1} \|x(t)\|$.

As in [5] we get an operator self-similar process. Such processes were first introduced by Lamperti [11] and play an important role in the context of long memory. Later operator self-similar processes were studied by Laha and Rohatgi [10]. In our case we get a self-similar process with values in a complex Hilbert space $\mathbb{H}$. With this in mind, we repeat the definition of self-similarity of Hilbert space-valued random sequences referring to Matache and Matache [13].

Definition 1.1. A stochastic process $\{Y(t) | t \geq 0\}$ on a Hilbert space $\mathbb{H}$ is called operator self-similar, if there exists a family $\{T(a) | a > 0\} \subset L(\mathbb{H})$, such that

$$\{Y(at) | t \geq 0\} \overset{f.d.d.}{=} \{T(a) Y(t) | t \geq 0\},$$

for each $a > 0$, where $\overset{f.d.d.}{=}$ denotes the equality of the finite-dimensional distributions.

The set $\{T(a) | a > 0\} \subset L(\mathbb{H})$ is also called scaling family of operators. If $T(a) = a^G I$, where G is a fixed scalar and I is the identity operator, the process is called G self-similar.