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Persistence probabilities for stationary increment processes

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Abstract

We study the persistence probability for processes with stationary increments. Our results apply to a number of examples: sums of stationary correlated random variables whose scaling limit is fractional Brownian motion; random walks in random sceneries; random processes in Brownian scenery; and the Matheron–de Marsily model in $\mathbb{Z}^2$ with random orientations of the horizontal layers. Using a new approach, strongly related to the study of the range, we obtain an upper bound of the optimal order in general and improved lower bounds (compared to previous literature) for many specific processes.

Keywords: Fractional Brownian motion; Random walk in random scenery; Persistence

1. Introduction

Persistence concerns the probability that a stochastic process has a long negative excursion. In this paper, we are concerned mainly with discrete-time processes. If $Z = (Z_n)_{n\geq 0}$ is a stochastic

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process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we study the rate of decay of the probability

$$\mathbb{P}\left(Z_n^* \leq a\right), \qquad \text{as } n \to +\infty, \quad \text{with} \quad Z_n^* := \max_{k=1,\ldots,n} Z_k\,,$$

for $a \in \mathbb{R}$. In many cases of interest, the above probability decreases polynomially, i.e. as $n^{-\theta+o(1)}$, and it is the first goal to find the persistence exponent θ. For a recent overview on this subject, we refer to the survey [3] and for the relevance in theoretical physics we recommend [8,23].

For random walks, Lévy processes, and a number of other Markov processes, persistence probabilities are well studied. E.g. for random walks, one can obtain the persistence probabilities with the help of the more general fluctuation theory. Since [27], persistence probabilities for intrinsically non-Markovian processes such as fractional Brownian motion are investigated. Methods based on exponential functionals have been developed to study persistence probabilities in such contexts (see e.g. [1,12,13]). In this paper, we present a new and simple approach adapted to the case of processes with stationary increments. This method is based on the study of the expectation of the number of elements in the range, provides results in non-Markovian contexts, and does not require the existence of exponential moments.

The purpose of this paper is to analyze the persistence probability for stationary increment discrete time processes, i.e. processes such that for any nonnegative integer k, $(Z_{n+k} - Z_k)_{n\geq 0} \stackrel{\mathcal{L}}{=} (Z_n)_{n\geq 0}$, where $\stackrel{\mathcal{L}}{=}$ means equality in law (note that this implies that $Z_0 = 0$). Under rather general assumptions, we prove that

$$\mathbb{P}\left(Z_n^* \leq a\right) \approx \mathbb{E}\left[Z_n^*\right]/n \qquad \text{as } n \to +\infty,$$

where $\approx$ means up to a multiplicative term in $n^{o(1)}$ (the multiplicative term is bounded by a constant for the upper bound and, for the lower bound, is larger than a function that is slowly varying at infinity). We emphasize the fact that we obtain the exact order when the increments are bounded and that we obtain estimates even if the increments admit no exponential moment. Further, we stress that $\mathbb{E}[Z_n^*]$ is in many cases a well-accessible quantity. All our conditions are formulated in terms of simple quantities of the process. Stationary increments are a feature shared by many stochastic processes that are important in theory and applications, and we shall treat here a number of examples.

The outline of the paper is as follows. Our main general results are stated and proved in Section 2. These general results are then applied to the following examples: Sums of stationary sequences and fractional Brownian motion are treated in Section 3, random walks in random scenery in Section 4, random processes in Brownian scenery in Section 5, and the Matheron–de Marsily model in Section 6.

2. General results for stationary increment processes

When dealing with persistence probabilities $\mathbb{P}\left(Z_n^* \leq a\right)$, a natural question is the choice of the level a. In most of the cases, going from a positive level to another positive level can be simply done by multiplying Z by some positive constant. The next technical lemma allows to pass from a negative level to a positive one and conversely, up to a change of the multiplicative constants in the bounds. This will be crucial as we often derive upper bounds for negative levels and lower bounds for positive levels, which have to be brought to matching.

Lemma 1. *Assume that there exists a sub-σ-algebra $\mathcal{F}_0$ of $\mathcal{F}$ such that, given $\mathcal{F}_0$, the increments of Z are positively associated and that their common conditional distribution is independent of*

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