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Conjugate processes: Theory and application to risk forecasting

Eduardo Horta*, Flavio Ziegelmann

Universidade Federal do Rio Grande do Sul, Department of Statistics, 9500 Bento Gonçalves Av., 43–111, Porto Alegre, RS, 91509-900, Brazil

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Abstract

Many dynamical phenomena display a cyclic behavior, in the sense that time can be partitioned into units within which distributional aspects of a process are homogeneous. In this paper, we introduce a class of models – called conjugate processes – allowing the sequence of marginal distributions of a cyclic, continuous-time process to evolve stochastically in time. The connection between the two processes is given by a fundamental compatibility equation. Key results include Laws of Large Numbers in the presented framework. We provide a constructive example which illustrates the theory, and give a statistical implementation to risk forecasting in financial data.

MSC: 60G57; 60G10; 62G99; 62M99

Keywords: Random measure; Covariance operator; Dimension reduction; Functional time series; High frequency financial data; Risk forecasting

1. Introduction

Many dynamical phenomena display a cyclic behavior, in the sense that time can be partitioned into units within which certain distributional aspects of a process are homogeneous. This idea is the starting point of the theory developed in Bosq [4], for instance. The standard probabilistic approach to modeling the evolution of a system over time usually begins with specification of a certain probability measure on the space of sample paths, induced by a family of

* Corresponding author.
E-mail address: eduardo.horta@ufrgs.br (E. Horta).

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finite-dimensional distributions. In this setting, consideration of conditional probabilities usually involves the notion of 'past information' as summarized by a filtering or the past trajectory of the process. We shall take a different approach, by introducing a latent process which permits us to interpret the cyclic character of a process in a conditional, distributional sense. We consider the following model. A sequence of random probability measures $\xi_0, \xi_1, \ldots, \xi_t, \ldots$ evolves stochastically in time. Associated to these probabilities is a continuous time, real-valued stochastic process $(X_\tau : \tau \geq 0)$ that satisfies the following condition, for each Borel set B in the real line,

$$\mathbb{P}[X_\tau \in B \mid \xi_0, \xi_1, \ldots] = \xi_t(B), \quad \tau \in [t, t+1). \tag{1}$$

We shall call each interval $[t, t+1)$ the tth *cycle*. Of course, Eq. (1) implies that, for $\tau \in [t, t+1)$, one has $\mathbb{P}[X_\tau \in B \mid \xi_0, \ldots, \xi_t] = \mathbb{P}[X_\tau \in B \mid \xi_t] = \xi_t(B)$. This can be interpreted as meaning that the process (X_τ) has marginal conditional distribution ξ_t during cycle t, and that past and future information about the $\xi_j's$ is to some extent irrelevant when ξ_t is given. Little further probabilistic structure is imposed on (X_τ). Notice however that the distribution of $(X_\tau : \tau \geq 0)$ is not entirely determined by (1). The model is potentially useful in situations where there is a natural notion of a cycle in the behavior of the process (X_τ), and where the main interest concerns statistical (i.e. distributional) aspects of the process, rather than 'sample-path' aspects, within each cycle. Possible applications include temperature measurements and intraday stock market return processes, the latter of which we illustrate below with a real data set. This model does have a Bayesian flavor, in that the distribution of the random variables X_τ are themselves random elements in a space of probability measures, but we shall not sail in this direction here. From now on we will go without saying that the index sets for t and τ are $0, 1, 2, \ldots$ and $\mathbb{R}^+$ respectively. A pair $(\xi_t,\ X_\tau)$, where (ξ_t) is a sequence of random probability measures, and (X_τ) is a process satisfying the compatibility condition (1), will be called a *conjugate process*. (ξ_t) is the *latent* (or *hidden*) *distribution process* and (X_τ) is the *observable process*. Notice that the probabilistic structure of the latent process (ξ_t) can be defined 'prior' to even mentioning the observable process (X_τ). In particular, the latter can be essentially anything as long as (1) holds.

In our model the evolution of (X_τ) over time is driven by the measure valued process (ξ_t). Consideration of random measures, together with the Hilbert space embedding introduced in Section 2.1, places our methodology in the realm of probability in function spaces, which has a long-standing tradition in the probability literature. The theory of probability measures on function spaces first rose from the need to interpret stochastic processes as random elements with values in spaces of functions, the original insight likely due to Wiener, who constructed a probability measure on the space of continuous functions – namely, Brownian motion – yet before Kolmogorov's axiomatization of probability theory. It eventually became clear that a convenient and quite general approach is to consider probability measures in metric spaces, as established for instance in the classic texts Billingsley [3] and Parthasarathy [25]. See also Van der Vaart and Wellner [27] for a modern account. A derived literature considers random elements (and hence probability measures) in Banach spaces, of which a very good exposition can be found in the classic texts Ledoux and Talagrand [20] and Vakhania et al. [28]. For stationary sequences and linear processes in Banach spaces, the monograph from Bosq [4] is a complete account.

The concept of random probability measures is very important in the theory of Bayesian nonparametrics — see Ghosh and Ramamoorthi [14]. Our approach however places us closer to the theory of inference on objects pertaining to function spaces, which in the statistics literature has come to be known as Functional Data Analysis (hereafter FDA) — see the cornerstone

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