SPA: 3146 Model 1 pp. 1–16 (col. fig: NIL)

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Stochastic Processes and their Applications xx (xxxx) xxx–xxx

stochastic processes and their applications

www.elsevier.com/locate/spa

On the regularity of American options with regime-switching uncertainty

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Received 31 January 2016; received in revised form 25 November 2016; accepted 7 June 2017
Available online xxxx

Abstract

We study the regularity of the stochastic representation of the solution of a class of initial–boundary value problems related to a regime-switching diffusion. This representation is related to the value function of a finite-horizon optimal stopping problem such as the price of an American-style option in finance. We show continuity and smoothness of the value function using coupling and time-change techniques. As an application, we find the minimal payoff scenario for the holder of an American-style option in the presence of regime-switching uncertainty under the assumption that the transition rates are known to lie within level-dependent compact sets.

MSC 2010: primary 60k37; 60G17; 60J70; secondary 93C30; 35B65; 91G80

Keywords: Regime-switching; Markov-modulated; Time-change; Coupling; American option; Initial–boundary value problem

1. Introduction

Let $B = (B_t)_{t\geq 0}$ be a Brownian motion and $Y = (Y_t)_{t\geq 0}$ be a continuous-time finite-state Markov chain, with respect to a common filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$, where $(\mathcal{F}_t)_{t\geq 0}$ satisfies the usual conditions. Note that Lemma 2.5 of [14] tells us that B and Y are independent.

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http://dx.doi.org/10.1016/j.spa.2017.06.007
0304-4149/

Please cite this article in press as: S.D. Jacka, A. Ocejo, On the regularity of American options with regime-switching uncertainty, Stochastic Processes and their Applications (2017), http://dx.doi.org/10.1016/j.spa.2017.06.007.

Let $\mathcal{S} = \{1, 2, \dots, m\}$ denote the state space of Y and $\pi = (\pi[i, j])$ its Q-matrix so that

$$\pi[i, j] \geq 0, \quad \text{for } i \neq j \text{ and } \quad \sum_{j=1}^{m} \pi[i, j] = 0 \quad \text{for } i = 1, 2, \dots, m.$$

Suppose that the process $X = (X_t)_{t \geq 0}$ obeys the stochastic differential equation with regime-switching

$$X_t = x + \int_0^t a(X_s, Y_s) dB_s + \int_0^t \mu(X_s, Y_s) ds, \qquad x \in \mathbb{R}, \tag{1}$$

where, for each $y \in \mathcal{S}$, $a(\cdot, y)$ and $\mu(\cdot, y)$ are, locally Lipschitz continuous on the state space of X and a is *positive*. Denote by $\mathbb{L}^\pi$ the operator related to the generator of the Markov process (X, Y), given by

$$\begin{aligned} \mathbb{L}^\pi w(x, y, t) = \frac{1}{2} a^2(x, y) w_{xx}(x, y, t) &+ \mu(x, y) w_x(x, y, t) - w_t(x, y, t) \\ &+ \sum_{y' \in \mathcal{S}, y' \neq y} [w(x, y', t) - w(x, y, t)] \pi[y, y']. \end{aligned} \tag{2}$$

For a given rate matrix π, consider the value of the optimal stopping problem with finite time horizon $T > 0$ and regime-switching associated with (X, Y):

$$v(x, y, t) = \sup_{\tau \leq t} E_{x,y}[e^{-\alpha\tau} g(X_\tau)], \qquad (x, y, t) \in \mathbb{R} \times \mathcal{S} \times [0, T], \tag{3}$$

where $\alpha \geq 0$ and $g : \mathbb{R} \to [0, \infty)$ is assumed to be a β-Hölder continuous function for some $0 < \beta \leq 1$. We write $E_{x,y}$ to denote the expectation conditioned on $(X_0 = x, Y_0 = y)$.

We are primarily interested in analytical properties of the value function in (3). In particular, we will show that for each $y \in \mathcal{S}$ the function $v(\cdot, y, \cdot) : \mathbb{R} \times [0, T] \to \mathbb{R}$ is $\beta/2$-Hölder continuous and v solves the initial–boundary value problem

$$\begin{aligned} (\mathbb{L}^\pi - \alpha) v(x, y, t) &= 0, && \text{in } \mathcal{C} \\ v(x, y, 0) &= g(x), && \text{in } \mathbb{R} \times \mathcal{S} \times \{0\} \\ v(x, y, t) &= g(x), && \text{on } \partial\mathcal{C} \end{aligned} \tag{4}$$

where $\mathcal{C} = \{(x, y, t) \in \mathbb{R} \times \mathcal{S} \times (0, T] : v(x, y, t) > g(x)\}$. This in turn yields that $v(\cdot, y, \cdot)$ is of the class $C^{2,1}$ in the set

$$\mathcal{C}_y = \{(x, t) \in \mathbb{R} \times (0, T] : \; v(x, y, t) > g(x)\}.$$

The stochastic representation of the solution of a problem of the form in (4) and in the setting where X is a diffusion without regime-switching is of course very well-known (see [13]) and the relation to an optimal stopping problem is standard [20,21]. This relationship allows the use of PDE methods to tackle the latter problem [21], such as finding a solution to an American option problem. However, in order to establish the desired connection, one typically requires continuity of the value function v. To this end, a number of subtle regularity conditions on the parameters of the problem must be imposed. In Section 2 we show that $v(\cdot, y, \cdot)$ is locally $\beta/2$-Hölder continuous (Theorem 2.4). Our results generalize those of Fleming and Soner [12] and Bayraktar, Song and Yang [4] in the context of optimal stopping (although they deal with combined control and stopping), since they assume uniformly Lipschitz coefficients and payoff and do not allow the jumps present in our model.

In the context of regime-switching diffusions, an early explicit example is Di Masi, Kabanov and Runggaldier [9], where they consider option pricing in an incomplete market with regime