SPA: 3151 | Model 1 | pp. 1–16 (col. fig: NIL)

ARTICLE IN PRESS

Available online at www.sciencedirect.com

ScienceDirect

Stochastic Processes and their Applications xx (xxxx) xxx–xxx

stochastic processes and their applications

www.elsevier.com/locate/spa

Time change equations for Lévy-type processes

Paul Krühner[a], Alexander Schnurr[b,c,*]

[a] Vienna University of Technology, FAM – Financial and Actuarial Mathematics, Wiedner Hauptstraße 8-10, AT–1040 Vienna, Austria

[b] Universität Siegen, Department Mathematik, Walter-Flex-Str. 3, DE–57068 Siegen, Germany

[c] Technische Universität Dortmund, Fakultät für Mathematik, Vogelpothsweg 87, DE–44227 Dortmund, Germany

Received 11 August 2015; received in revised form 6 December 2016; accepted 20 June 2017

Available online xxxx

Abstract

We consider time change equations for Lévy-type processes. In this context we generalize the results of Böttcher et al. (2013) significantly. Namely, we are able to incorporate measurable instead of continuous multipliers. This opens a gate to find whole classes of symbols for which corresponding processes do exist. In order to establish our results we carefully analyze the connection between time change equations and classical initial value problems. This relationship allows us to transfer well-known results from this classical subject of pure mathematics into the theory of stochastic processes. On the way to prove our main theorem we establish generalizations of results on paths of Lévy-type processes.

MSC 2010: 60J75; 45G10; 60G17

Keywords: Lévy-type process; Symbol; Random time change; Multiplicative perturbation

1. Introduction

The study of multiplicative perturbation has started with early papers like Dorroh [7] and the more general versions by Gustafson and Lumer [13], see also Jacob [14]. Dorroh has focused on the very relevant contraction semigroups on continuous function spaces perturbed with a multiplier where the multiplying function is continuous, bounded and strictly positive.

* Corresponding author at: Universität Siegen, Department Mathematik, Walter-Flex-Str. 3, DE–57068 Siegen, Germany.

E-mail addresses: paulkrue@fam.tuwien.ac.at (P. Krühner), schnurr@mathematik.uni-siegen.de (A. Schnurr).

URLs: https://fam.tuwien.ac.at/∼paulkrue/ (P. Krühner), https://www.uni-siegen.de/fb6/src/schnurr/ (A. Schnurr).

http://dx.doi.org/10.1016/j.spa.2017.06.011

0304-4149/

Please cite this article in press as: P. Krühner, A. Schnurr, Time change equations for Lévy-type processes, Stochastic Processes and their Applications (2017), http://dx.doi.org/10.1016/j.spa.2017.06.011.

Pre-dating the paper of Dorroh, Volkonskii [25] has found a path transformation between Brownian motion and continuous sample path Markov processes on the real line. The similar transformation used by Lamperti [20] relates positive self-similar processes and Lévy processes. The generalization of this path transformation connects the analytic multiplicative perturbation theory with pathwise transformation of the corresponding stochastic processes. Ethier and Kurtz [9, Section 6] have investigated in their book the connection of the stochastic transform to the analytic multiplicative perturbation. In the present work we specialize it to Lévy-type processes which are characterized by their symbol q, cf. Jacob and Schilling [15] which can be seen as an encoding of the state based characteristics $(b(x), c(x), F(x,\cdot))_{x\in E}$ via $q(x,u) = i\langle u, b(x)\rangle - \frac{1}{2}\langle c(x)u, u\rangle + \int_{\mathbb{R}^d}\left(e^{i\langle u,y\rangle} - 1 - i\langle u, \chi(y)\rangle\right) F(x, dy)$, cf. Proposition 2.7. Böttcher, Schilling and Wang [3] summarize that if $(q(x,u))_{x\in E, u\in\mathbb{R}^d}$ is a symbol that belongs to a Markov process and $g : E \to \mathbb{R}$ is continuous, bounded and strictly positive, then $(g(x)q(x,u))_{x\in E,u\in\mathbb{R}^d}$ belongs to a Markov process as well. This is essentially the translation of Dorroh's result to the stochastic setup. In the book of Ethier and Kurtz [9, Section 6] this approach is more general in the sense that they do allow for the multiplying function g to be only measurable and non-negative, however, the technical condition in their Theorem 1.1 cannot be verified easily from the symbol and the multiplying function. Engelbert and Schmidt [8] optimize the original approach of Volkonskii [25] and find exact conditions under which a multiplicative perturbed Brownian motion gives rise to a strong Markov process. Their approach is based on detailed knowledge of the Brownian local time or in some sense on path properties of the Wiener process. Path properties of Lévy-type processes have been studied in the paper of Schnurr [24] and are utilized herein to improve the result in Böttcher et al. [3] in two ways. First, we do allow that the multiplying function hits zero and, second, we allow for measurable instead of continuous multiplying functions. For a continuous multiplier we essentially need that the function does not grow too quickly near its zeros in order to ensure that these points become absorbing states. This, allows to adapt the arguments in Böttcher et al. [3] or Ethier and Kurtz [9].

Applications of random time changes include the seminal work of Volkonskii [25], the simple necessary and sufficient condition in Engelbert and Schmidt [8], the work of Lamperti [20] to identify self-similar Markov process, see also Döring [5] and application to affine processes by Kallsen [18] and Gabrielli and Teichmann [12]. Time change techniques are also relevant for applications in finance, see e.g. [6]. Let us give a brief motivation in this context: if a stock price is modeled by $S(t) = \exp(X(t))$ where X is a Lévy(-type) process, then it might be additionally desirable to account for the effect that decreased stock prices lead to a rise in trading activity. If the stock price is modeled instead by $S(t) = \exp(Z(t))$ where $Z(t) = X(\int_0^t g(Z(s))ds)$, then Z has higher activity when g is larger than 1, normal activity where $g = 1$ and decreased activity where g is smaller than one which easily allows to model various activity regions.

The question for which symbols $q : E \times \mathbb{R}^d \to \mathbb{C}$ (or classes of symbols), there exists a corresponding stochastic process is a vital part of ongoing research. Compare in this context: Jacob and Schilling [15] and Böttcher et al. [3]. Techniques in order to establish existence results include approaches via Dirichlet forms, the Hille–Yosida–Ray theorem and solutions to the martingale problem. Here, we contribute to this part of the theory using an approach via time changes. If one can prove by any of the above techniques that for the symbol q, there exists a corresponding process X then we get for the whole class of symbols which can be written as $g(x)q(x,u)$, with g as described below, that corresponding processes do exist.

The time change equations which are used in the present article are a certain kind of random time changes. They have to be distinguished from other random time changes like Bochner's subordination. In this latter concept an independent increasing process $L(u)$ serves as a new time scale of the process X, that is, $(X(L(u)), u \geq 0)$ is being considered (cf. [23]).

Please cite this article in press as: P. Krühner, A. Schnurr, Time change equations for Lévy-type processes, Stochastic Processes and their Applications (2017), http://dx.doi.org/10.1016/j.spa.2017.06.011.