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Central limit theorem for functionals of a generalized self-similar Gaussian process

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Abstract

We consider a class of self-similar, continuous Gaussian processes that do not necessarily have stationary increments. We prove a version of the Breuer–Major theorem for this class, that is, subject to conditions on the covariance function, a generic functional of the process increments converges in law to a Gaussian random variable. The proof is based on the Fourth Moment Theorem. We give examples of five non-stationary processes that satisfy these conditions.
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1. Introduction

We consider a centered Gaussian process $X = \{X_t, t \geq 0\}$ that is self-similar of order $\beta \in (0, 1)$. That is, the process $\{a^{-\beta} X_{at}, t \geq 0\}$ has the same distribution as the process X for any $a > 0$. Consider the function $\phi : [1, \infty) \to \mathbb{R}$ given by

$$\phi(x) = \mathbb{E}[X_1 X_x]. \tag{1}$$

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This function characterizes the *covariance function*. Indeed, for $0 < s \le t$, we have

$$R(s,t) = \mathbb{E}\left[X_s X_t\right] = s^{2\beta}\mathbb{E}\left[X_1 X_{t/s}\right] = s^{2\beta}\phi\left(\frac{t}{s}\right).$$

The best known self-similar Gaussian process is the fractional Brownian motion (fBm), where

$$R(s,t) = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right),$$

and the self-similarity exponent β is the Hurst parameter $H \in (0,1)$. For the fBm,

$$\phi(x) = \frac{1}{2}\left(1 + x^{2H} - (x-1)^{2H}\right), \qquad x \ge 1.$$

We will impose conditions on the function ϕ such that $\mathbb{E}\left[(X_{t+s} - X_t)^2\right] \sim s^\alpha$ as $s \to 0$, for some constant $\alpha \in (0, 2\beta]$ that we call the *increment exponent*. For example, $\alpha = 2H$ for the fBm, since $\mathbb{E}\left[(X_{t+s} - X_t)^2\right] = s^{2H}$. However, there are examples where $\alpha < 2\beta$.

Our goal in this paper is to identify a set of conditions on α, β and ϕ, such that we can establish a central limit theorem for functionals of the increments of X. More precisely, let $\gamma = \mathcal{N}(0,1)$ and consider a function $f \in L^2(\mathbb{R}, \gamma)$, which has an expansion of the form

$$f(x) = \sum_{q=d}^{\infty} c_q H_q(x), \tag{2}$$

where $d \ge 1$, $c_d \ne 0$, and $H_q(x)$ is the qth Hermite polynomial, defined as

$$H_q(x) = (-1)^q e^{\frac{x^2}{2}} \frac{d^q}{dx^q} e^{-\frac{x^2}{2}}.$$

The index d is called the *Hermite rank* of f.

For integers $n \ge 2$ and $j \ge 0$ define

$$\Delta X_{\frac{j}{n}} = X_{\frac{j+1}{n}} - X_{\frac{j}{n}} \quad \text{and} \quad Y_{j,n} = \frac{\Delta X_{\frac{j}{n}}}{\|\Delta X_{\frac{j}{n}}\|_{L^2(\Omega)}}.$$

We consider the stochastic process defined by

$$F_n(t) = \frac{1}{\sqrt{n}} \sum_{j=0}^{\lfloor nt \rfloor - 1} f(Y_{j,n}), \quad \lfloor nt \rfloor \ge 1, \tag{3}$$

and $F_n(t) = 0$ if $\lfloor nt \rfloor < 1$. It is well known that if the process X has stationary increments, the convergence to a normal law for the sequence of random variables $F_n(t)$ can be deduced from the following central limit theorem proved by Breuer and Major in [3].

Theorem 1.1. *Suppose $\{Y_j, j \ge 1\}$ is a centered stationary Gaussian sequence with unit variance, and denote by $\rho(k) = \mathbb{E}(Y_n Y_{n+k})$ its covariance function. Consider a function $f \in L^2(\mathbb{R}, \gamma)$ with Hermite rank $d \ge 1$. Then the functional*

$$V_n = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} f(Y_j) \tag{4}$$

converges in distribution to a normal law $\mathcal{N}(0, \sigma^2)$ as n tends to infinity, provided $\sum_{k\in\mathbb{Z}} |\rho(k)|^d < \infty$, and in this case $\sigma^2 = \sum_{q=d}^{\infty} q! c_q^2 \sum_{k\in\mathbb{Z}} \rho(k)^q$.

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