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The Hausdorff dimension of multivariate operator-self-similar Gaussian random fields

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Abstract

Let $\{X(t) : t \in \mathbb{R}^d\}$ be a multivariate operator-self-similar random field with values in $\mathbb{R}^m$. Such fields were introduced in [22] and satisfy the scaling property $\{X(c^E t) : t \in \mathbb{R}^d\} \stackrel{\mathrm{d}}{=} \{c^D X(t) : t \in \mathbb{R}^d\}$ for all $c > 0$, where E is a $d \times d$ real matrix and D is an $m \times m$ real matrix. We solve an open problem in [22] by calculating the Hausdorff dimension of the range and graph of a trajectory over the unit cube $K = [0, 1]^d$ in the Gaussian case. In particular, we enlighten the property that the Hausdorff dimension is determined by the real parts of the eigenvalues of E and D as well as the multiplicity of the eigenvalues of E and D.

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1. Introduction

In this paper we consider multivariate operator-self-similar random fields as introduced in [22]. More precisely, let $E \in \mathbb{R}^{d\times d}$ and $D \in \mathbb{R}^{m\times m}$ be real matrices with positive real parts of their eigenvalues. A random field $\{X(t) : t \in \mathbb{R}^d\}$ with values in $\mathbb{R}^m$ is called multivariate operator-self-similar for E and D or (E, D)-operator-self-similar if

$$\{X(c^E x) : x \in \mathbb{R}^d\} \stackrel{\mathrm{d}}{=} \{c^D X(x) : x \in \mathbb{R}^d\} \quad \text{for all } c > 0, \tag{1.1}$$

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where $\stackrel{\mathrm{d}}{=}$ means equality of all finite-dimensional marginal distributions and, as usual, $c^A = \exp(A \log c) = \sum_{k=0}^{\infty} \frac{(\log c)^k}{k!} A^k$ is the matrix exponential for every matrix A. In the literature $\mathbb{R}^d$ is usually referred to as the time-domain, $\mathbb{R}^m$ as the state space and X is called a (d, m)-random field. Furthermore, E is called the time-variable scaling exponent and D the state space scaling operator.

These fields can be seen as a generalization of both operator-self-similar processes (see [16,19,29]) and operator scaling random fields (see [6,8]). Let us recall that a stochastic process $\{X(t) : t \in \mathbb{R}\}$ with values in $\mathbb{R}^m$ is called operator-self-similar if

$$\{X(ct) : t \in \mathbb{R}\} \stackrel{\mathrm{d}}{=} \{c^D X(t) : t \in \mathbb{R}\} \quad \text{for all } c > 0,$$

whereas a scalar valued random field $\{Y(t) : t \in \mathbb{R}^d\}$ is said to be operator scaling for E and some $H > 0$ if

$$\{Y(c^E t) : t \in \mathbb{R}^d\} \stackrel{\mathrm{d}}{=} \{c^H Y(t) : t \in \mathbb{R}^d\} \quad \text{for all } c > 0.$$

Operator-self-similar processes have been studied extensively during the past decades due to their theoretical importance and are also used in various applications such as physics, engineering, biology, mathematical finance, just to mention a few (see e.g. [1,10,20,28,31]). Several authors proposed to apply operator scaling random fields for modeling phenomena in spatial statistics, hydrology and image processing (see e.g. [5,9,12]).

A very important class of operator-self-similar random fields is given by Gaussian random fields and especially by the so-called operator-fractional Brownian motion B_D with state space scaling operator D (see [24]), also known as operator-fractional Brownian field [4,13]. The random field B_D fulfills the self-similarity relation

$$\{B_D(ct) : t \in \mathbb{R}^d\} \stackrel{\mathrm{d}}{=} \{c^D B_D(t) : t \in \mathbb{R}^d\}$$

and has stationary increments, i.e. it satisfies

$$\{B_D(t+h) - B_D(h) : t \in \mathbb{R}^d\} \stackrel{\mathrm{d}}{=} \{B_D(t) : t \in \mathbb{R}^d\}$$

for any $h \in \mathbb{R}^d$. We remark that Mason and Xiao [24] studied several sample path properties of B_D including fractal dimensions of the range and the graph of B_D. More precisely, for any arbitrary Borel set $F \subset \mathbb{R}^d$, under some additional assumptions (see [24, Theorem 4.1]), they showed that a.s. the Hausdorff dimensions of the range and graph are given by

$$\dim_{\mathcal{H}} B_D(F) = \min\Big\{m, \Big(\dim_{\mathcal{H}} F + \sum_{i=1}^{j} (\lambda_j - \lambda_i)\Big)\lambda_j^{-1}, j = 1, \ldots, m\Big\}, \tag{1.2}$$

$$\dim_{\mathcal{H}} \mathrm{Gr}\, B_D(F) = \begin{cases} \dim_{\mathcal{H}} B_D(F) & \text{if } \dim_{\mathcal{H}} F \le \sum_{i=1}^{m} \lambda_i, \\ \dim_{\mathcal{H}} F + \sum_{i=1}^{m} (1 - \lambda_i) & \text{if } \dim_{\mathcal{H}} F > \sum_{i=1}^{m} \lambda_i, \end{cases} \tag{1.3}$$

where $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_m < 1$ are the real parts of the eigenvalues of D. In particular, if $F = [0, 1]^d$ they obtain that a.s.

$$\dim_{\mathcal{H}} B_D(F) = \min\Big\{m, \Big(d + \sum_{i=1}^{j} (\lambda_j - \lambda_i)\Big)\lambda_j^{-1}, j = 1, \ldots, m\Big\}, \tag{1.4}$$