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The asymptotic smile of a multiscaling stochastic volatility model

Francesco Caravenna[a,*], Jacopo Corbetta[b]

[a] *Dipartimento di Matematica e Applicazioni, Università degli Studi di Milano-Bicocca, via Cozzi 55, I-20125 Milano, Italy*

[b] *École des Ponts - ParisTech, CERMICS, 6 et 8 avenue Blaise Pascal, 77420 Champs sur Marne, France*

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Abstract

We consider a stochastic volatility model which captures relevant stylized facts of financial series, including the multi-scaling of moments. The volatility evolves according to a generalized Ornstein–Uhlenbeck processes with super-linear mean reversion.

Using large deviations techniques, we determine the asymptotic shape of the implied volatility surface in any regime of small maturity $t \to 0$ or extreme log-strike $|\kappa| \to \infty$ (with bounded maturity). Even if the price has continuous paths, out-of-the-money implied volatility diverges for small maturity, producing a very pronounced smile.

MSC: primary 60F10; secondary 91B25, 60G44; 60G44

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1. Introduction

The evolution of the price $(S_t)_{t\geq 0}$ of an asset is often described by a stochastic volatility model $\mathrm{d}S_t = S_t(\mu\,\mathrm{d}t + \sigma_t\,\mathrm{d}B_t)$, where $(B_t)_{t\geq 0}$ is a standard Brownian motion and $(\sigma_t)_{t\geq 0}$ is a stochastic process. A popular choice for $(\sigma_t)_{t\geq 0}$ is a process of Ornstein–Uhlenbeck type:

$$\mathrm{d}\sigma_t^2 = -c\,(\sigma_t^2)^\gamma\,\mathrm{d}t + \mathrm{d}L_t, \tag{1.1}$$

* Corresponding author.
E-mail addresses: francesco.caravenna@unimib.it (F. Caravenna), jacopo.corbetta@enpc.fr (J. Corbetta).

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where $(L_t)_{t\geq 0}$ is a subordinator (i.e. a non-decreasing Lévy process) and $c, \gamma \in (0, \infty)$ are parameters, the usual choice being the case $\gamma = 1$ when the mean reversion is linear, cf. [3]. This class of models is rich enough to reproduce many empirically observed stylized facts, including heavy tails in the distribution of S_t and clustering of volatility.

Another remarkable stylized fact is the so-called *multi-scaling of moments* [10,11,14]. This refers to the fact that $\mathrm{E}[|S_{t+h} - S_t|^q] \approx h^{A(q)}$ as $h \to 0$, where the scaling exponent is diffusive only up to a finite threshold, i.e. $A(q) = q/2$ for $q < q^*$, while for $q > q^*$ an anomalous scaling $A(q) < q/2$ is observed. Interestingly, it was recently proved in [8] that a stochastic volatility model with σ_t as in (1.1) does not exhibit multi-scaling of moments in the linear case $\gamma = 1$; however, *multi-scaling of moments does occur in the super-linear case* $\gamma > 1$, if the Lévy measure of $(L_t)_{t\geq 0}$ has a polynomial tail at infinity.

It is natural to ask how stochastic volatility models with σ_t as in (1.1) behave with respect to pricing, when $\gamma > 1$. This is a non-trivial problem, because the moment generating function of S_t typically admits no closed form outside the linear case $\gamma = 1$. However, there is a special limiting case which is analytically more tractable, defined as follows.

Consider a subordinator with finite activity: $L_t = \sum_{k=1}^{N_t} J_k$, where $(N_t)_{t\geq 0}$ is a Poisson process and $(J_k)_{k\in\mathbb{N}}$ are i.i.d. non-negative random variables. In this case Eq. (1.1) can be solved pathwise, i.e. for any fixed realization of $(L_t)_{t\geq 0}$, because between jump times of the Poisson process $(N_t)_{t\geq 0}$ it reduces to the ordinary differential equation

$$\mathrm{d}(\sigma_t^2) = -c\,(\sigma_t^2)^\gamma\,\mathrm{d}t, \tag{1.2}$$

which admits explicit solutions. The point is that, when $\gamma > 1$, one can *let the jump size diverge* $J_k \to \infty$ and $(\sigma_t)_{t\geq 0}$ converges to a well-defined limiting process, which explodes at the jump times of the Poisson process and solves (1.2) between them (see Fig. 1(a)). For $\gamma > 2$, this limiting process $(\sigma_t)_{t\geq 0}$ has square-integrable paths and can therefore be used to define a stochastic volatility model.

In this paper we focus on this stochastic volatility model, which was introduced in [2] (in a more direct way, see Section 2) and was shown to display several interesting features, including multi-scaling of moments, clustering of volatility and the crossover in the log-return distribution from power-law (small time) to Gaussian (large time). We are interested in the price of European option and in the corresponding implied volatility.

We stress that, besides its own interest, our model retains a close link with the general class of stochastic volatility models $\mathrm{d}S_t = S_t(\mu\,\mathrm{d}t + \sigma_t\,\mathrm{d}B_t)$ with σ_t as in (1.1) with $\gamma > 2$. For instance, option price and implied volatility of our model *provide an upper bound for all models in this class with a finite activity subordinator* $(L_t)_{t\geq 0}$ (see Section 3.3).

Our main results are sharp estimates for the tail decay of the log-return distribution (Theorem 4.1), which yield explicit asymptotic formulas for the price of European options (Theorem 4.3) and for the corresponding *implied volatility surface* (Theorem 3.2). Let us summarize some of the highlights, referring to Section 3.4 for a more detailed discussion.

- We allow for any regime of either extreme log-strike $|\kappa| \to \infty$ (with arbitrary bounded maturity t, possibly varying with κ) or small maturity $t \downarrow 0$ (with arbitrary log-strike κ, possibly varying with t). This flexibility yields uniform estimates for the implied volatility surface $\sigma_{\mathrm{imp}}(\kappa, t)$ in *open regions* of the plane (κ, t), cf. Corollary 3.6.
- We show that out-of-the-money implied volatility *diverges* for small maturity, i.e. $\sigma_{\mathrm{imp}}(\kappa, t) \to \infty$ as $t \downarrow 0$ for any $\kappa \neq 0$, while $\sigma_{\mathrm{imp}}(0, t) \to \sigma_0 < \infty$ (see Fig. 2). This shows that stochastic volatility models without jumps in the price can produce very steep skews for the