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Limit theorems for critical first-passage percolation on the triangular lattice

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Abstract

Consider (independent) first-passage percolation on the sites of the triangular lattice $\mathbb{T}$ embedded in $\mathbb{C}$. Denote the passage time of the site v in $\mathbb{T}$ by $t(v)$, and assume that $P(t(v) = 0) = P(t(v) = 1) = 1/2$. Denote by $b_{0,n}$ the passage time from 0 to the halfplane $\{v \in \mathbb{T} : \mathrm{Re}(v) \geq n\}$, and by $T(0, nu)$ the passage time from 0 to the nearest site to nu, where $|u| = 1$. We prove that as $n \to \infty$, $b_{0,n}/\log n \to 1/(2\sqrt{3}\pi)$ a.s., $E[b_{0,n}]/\log n \to 1/(2\sqrt{3}\pi)$ and $\mathrm{Var}[b_{0,n}]/\log n \to 2/(3\sqrt{3}\pi) - 1/(2\pi^2)$; $T(0, nu)/\log n \to 1/(\sqrt{3}\pi)$ in probability but not a.s., $E[T(0, nu)]/\log n \to 1/(\sqrt{3}\pi)$ and $\mathrm{Var}[T(0, nu)]/\log n \to 4/(3\sqrt{3}\pi) - 1/\pi^2$. This answers a question of Kesten and Zhang (1997) and improves our previous work (2014). From this result, we derive an explicit form of the central limit theorem for $b_{0,n}$ and $T(0, nu)$. A key ingredient for the proof is the moment generating function of the conformal radii for conformal loop ensemble CLE_6, given by Schramm et al. (2009).

MSC: 60K35; 82B43

Keywords: Critical percolation; First-passage percolation; Scaling limit; Conformal loop ensemble; Law of large numbers; Central limit theorem

1. Introduction

First-passage percolation (FPP) was introduced by Hammersley and Welsh in 1965 as a model of fluid flow through a random medium. We refer the reader to the recent surveys [2,9]. In this paper, we continue our study of critical FPP on the triangular lattice $\mathbb{T}$, initiated in [25]. We focus

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on this particular lattice because our proof relies on the existence of the scaling limit of critical site percolation on $\mathbb{T}$ (see [4,19]), and this result has not been proved for other planar percolation processes. For recent progress on general planar critical FPP, see [7].

Let $\mathbb{T} = (\mathbb{V}, \mathbb{E})$ denote the triangular lattice, where $\mathbb{V} := \{x + ye^{\pi i/3} \in \mathbb{C} : x, y \in \mathbb{Z}\}$ is the set of sites, and $\mathbb{E}$ is the set of bonds, connecting adjacent sites. Let $\{t(v) : v \in \mathbb{V}\}$ be an i.i.d. family of Bernoulli random variables:

$$P[t(v) = 0] = P[t(v) = 1] = \frac{1}{2}.$$

We call this model **Bernoulli critical FPP** on $\mathbb{T}$, and denote by P its probability measure. Note that we can view this model as critical site percolation on $\mathbb{T}$ (see e.g. [15,23] for background on two-dimensional critical percolation). We usually represent it as a random coloring of the faces of the dual hexagonal lattice $\mathbb{H}$, each face centered at $v \in \mathbb{V}$ being blue ($t(v) = 0$) or yellow ($t(v) = 1$) with probability $1/2$ independently of the others. Sometimes we view the site v as the hexagon in $\mathbb{H}$ centered at v.

A **path** is a sequence $v_0, \ldots, v_n$ of distinct sites of $\mathbb{T}$ such that v_{k-1} and v_k are neighbors for all $k = 1, \ldots, n$. For a path γ, we define its passage time as $T(\gamma) := \sum_{v \in \gamma} t(v)$. The **first-passage time** between two site sets A, B is defined as

$$T(A, B) := \inf\{T(\gamma) : \gamma \text{ is a path from a site in } A \text{ to a site in } B\}.$$

For any $u \in \mathbb{C}$ with $|u| = 1$, denote by $T(0, nu)$ the first-passage time from 0 to the nearest site in $\mathbb{V}$ to nu (if there is more than one such site, we choose a unique one by some deterministic method). Denote by $b_{0,n}$ the first-passage time from 0 to the halfplane $\{v \in \mathbb{V} : \Re(v) \geq n\}$.

Our main theorem below answers a question proposed by Kesten and Zhang (see (1.10) and (1.11) in [11]):

Theorem 1.1.

$$\lim_{n\to\infty} \frac{b_{0,n}}{\log n} = \frac{1}{2\sqrt{3}\pi} \quad a.s., \tag{1}$$

$$\lim_{n\to\infty} \frac{E[b_{0,n}]}{\log n} = \frac{1}{2\sqrt{3}\pi}, \tag{2}$$

$$\lim_{n\to\infty} \frac{\mathrm{Var}[b_{0,n}]}{\log n} = \frac{2}{3\sqrt{3}\pi} - \frac{1}{2\pi^2}. \tag{3}$$

For each $u \in \mathbb{C}$ with $|u| = 1$,

$$\lim_{n\to\infty} \frac{T(0, nu)}{\log n} = \frac{1}{\sqrt{3}\pi} \quad \text{in probability but not a.s.,} \tag{4}$$

$$\lim_{n\to\infty} \frac{E[T(0, nu)]}{\log n} = \frac{1}{\sqrt{3}\pi}, \tag{5}$$

$$\lim_{n\to\infty} \frac{\mathrm{Var}[T(0, nu)]}{\log n} = \frac{4}{3\sqrt{3}\pi} - \frac{1}{\pi^2}. \tag{6}$$

Remark. In [25], we proved the law of large numbers for $b_{0,n}$ and $T(0, nu)$, but could not give exact values of the limits. The proof in that paper relies on the subadditive ergodic theorem, which is a nice tool to show the existence of the limit but gives no insight for the exact value of the limit.