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Domain and range symmetries of operator fractional Brownian fields[☆]

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Abstract

An operator fractional Brownian field (OFBF) is a Gaussian, stationary increment $\mathbb{R}^n$-valued random field on $\mathbb{R}^m$ that satisfies the operator self-similarity property $\{X(c^E t)\}_{t\in\mathbb{R}^m} \stackrel{\mathcal{L}}{=} \{c^H X(t)\}_{t\in\mathbb{R}^m}$, $c > 0$, for two matrix exponents (E, H). In this paper, we characterize the domain and range symmetries of OFBF, respectively, as maximal groups with respect to equivalence classes generated by orbits and, based on a new anisotropic polar-harmonizable representation of OFBF, as intersections of centralizers. We also describe the sets of possible pairs of domain and range symmetry groups in dimensions $(m, 1)$ and $(2, 2)$.

MSC: primary 60G18; 60G15

Keywords: Operator fractional Brownian field; Symmetry group; Anisotropy; Operator scaling; Operator self-similarity; Long range dependence

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1. Introduction

A random vector is called *full* if its distribution is not supported on a lower dimensional hyperplane. A random field $X = \{X(t)\}_{t\in\mathbb{R}^m}$ with values in $\mathbb{R}^n$ is called *proper* if $X(t)$ is full for all $t \neq 0$. A linear operator P on $\mathbb{R}^m$ is called a projection if $P^2 = P$. Any nontrivial projection $P \neq I$ maps $\mathbb{R}^m$ onto a lower dimensional subspace. We say that a random field X is *degenerate* if there exists a nontrivial projection P such that $X(t) = X(Pt)$ for all $t \in \mathbb{R}^m$. We say that X is *stochastically continuous* if $X(t_n) \to X(t)$ in probability whenever $t_n \to t$. A proper, nondegenerate, and stochastically continuous random vector field X is called *operator self-similar* (o.s.s.) if

$$\{X(c^E t)\}_{t\in\mathbb{R}^m} \stackrel{\mathcal{L}}{=} \{c^H X(t)\}_{t\in\mathbb{R}^m} \quad \text{for all } c > 0. \tag{1.1}$$

In (1.1), $\stackrel{\mathcal{L}}{=}$ indicates equality of finite-dimensional distributions, $E \in M(m,\mathbb{R})$ and $H \in M(n,\mathbb{R})$, where $M(p,\mathbb{R})$ represents the space of real-valued $p \times p$ matrices, and $c^M = \exp(M(\log c)) = \sum_{k=0}^{\infty}(M \log c)^k/k!$ for a square matrix M. For a univariate stochastic process (namely, $(m,n) = (1,1)$), the relation (1.1) is called self-similarity (see, for example, [17,35]).

An *operator fractional Brownian field* (OFBF, in short) is an $\mathbb{R}^n$-valued random field $X = \{X(t)\}_{t\in\mathbb{R}^m}$ satisfying the following three properties: (i) it is Gaussian with mean zero; (ii) it is o.s.s.; (iii) it has stationary increments, that is, for any $h \in \mathbb{R}^m$, $\{X(t+h) - X(h)\}_{t\in\mathbb{R}^m} \stackrel{\mathcal{L}}{=} \{X(t) - X(0)\}_{t\in\mathbb{R}^m}$. When $(m,n) = (1,1)$, OFBF is the celebrated fractional Brownian motion, widely used in applications due to the long-range dependence property of its increments (see [34,16]). When $m = 1, n \geq 1$, OFBF is known as operator fractional Brownian motion (OFBM).

The theory of o.s.s. stochastic processes ($m = 1$, $n \geq 1$) was developed by Laha and Rohatgi [24] and Hudson and Mason [21], see also Chapter 11 in [29]. OFBM was studied by Didier and Pipiras [13] (see also [1,33,22,23] on the related subject of multivariate long-range dependent time series). For scalar fields (namely, $m \geq 1$, $n = 1$), the analogues of fractional Brownian motion and fractional stable motion were studied in depth by Biermé et al. [7], with related work and applications found in [4,8,25,6,5,19,9,10,28,15,32]. Li and Xiao [26] proved important results on o.s.s. random vector fields. Baek et al. [2] bridged the gap between harmonizable and moving average integral representations for OFBF.

The *domain and range symmetries* of a proper, nondegenerate random field X starting at zero are defined by

$$\begin{aligned} G_1^{\mathrm{dom}}(X) &:= \Big\{A \in GL(m,\mathbb{R}) : \{X(At)\} \stackrel{\mathcal{L}}{=} \{X(t)\}\Big\},\\ G_1^{\mathrm{ran}}(X) &:= \Big\{B \in GL(n,\mathbb{R}) : \{BX(t)\} \stackrel{\mathcal{L}}{=} \{X(t)\}\Big\}, \end{aligned} \tag{1.2}$$

where $GL(k,\mathbb{R})$ denotes the general linear group of invertible $k \times k$ matrices. Cohen et al. [11] and Didier and Pipiras [14], respectively, characterized the range symmetries of operator stable Lévy processes and OFBM.

Symmetry is an important modeling consideration, and a useful guide to model selection (see [27] on Markov processes and [30] on measures). In particular, the interest in the study of symmetries is tightly connected to two major themes: (a) anisotropy, i.e., when $G_1^{\mathrm{dom}}(X)$ is not the orthogonal group, and its applications in several fields such as bone radiographic imaging and hydrology; and (b) the parametric identification of operator scaling laws, which depends on both $G_1^{\mathrm{dom}}(X)$ and $G_1^{\mathrm{ran}}(X)$. The latter theme is treated in detail for general o.s.s. random fields in the related paper [12]. In regard to the former, note that the term "anisotropy", like "nonlinear"