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Distribution dependent SDEs for Landau type equations☆

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Abstract

The distribution dependent stochastic differential equations (DDSDEs) describe stochastic systems whose evolution is determined by both the microcosmic site and the macrocosmic distribution of the particle. The density function associated with a DDSDE solves a nonlinear PDE. Due to the distribution dependence, some standard techniques developed for SDEs do not apply. By iterating in distributions, a strong solution is constructed using SDEs with control. By proving the uniqueness, the distribution of solutions is identified with a nonlinear semigroup P_t^* on the space of probability measures. The exponential contraction as well as Harnack inequalities and applications are investigated for the nonlinear semigroup P_t^* using coupling by change of measures. The main results are illustrated by homogeneous Landau equations.

MSC: 60J75; 47G20; 60G52

Keywords: Distribution dependent SDEs; Homogeneous Landau equation; Wasserstein distance; Exponential convergence; Harnack inequality

1. Introduction

A fundamental application of the Itô SDE is to solve Kolmogorov's problem [15] of determining Markov processes whose distribution density satisfies the Fokker–Planck–Kolmogorov equation. Let W_t be the d-dimensional Brownian motion on a complete probability space with

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natural filtration $(\Omega, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$, and let $b: \mathbb{R}^d \to \mathbb{R}^d$; $\sigma: \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ be measurable. Then the distribution density of the solution to the SDE

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t \tag{1.1}$$

satisfies the parabolic equation

$$\partial_t f_t = \frac{1}{2}\sum_{i,j=1}^d \partial_i\partial_j\big\{(\sigma\sigma^*)_{ij} f_t\big\} - \sum_{i=1}^d \partial_i\{b_i f_t\}, \tag{1.2}$$

which describes the time evolution of the probability density function of the velocity of a particle under the influence of drag forces and random forces. If b and σ are "almost" locally Lipschitzian, then the SDE (1.1) has a unique strong solution up to life time (c.f. [8]). When σ is invertible (i.e. the SDE is non-degenerate), this condition has been largely weakened as $|b| + |\nabla\sigma| \in L^p_{loc}(\mathrm{d}x)$ for some $p > d$, see [28] and references within.

When coefficients σ and b also depend on the distribution of the solution, the SDE is called distribution dependent. This type SDEs often arise from mathematical physics, see for instance [10] for distribution dependent SDEs of jump type describing the Boltzmann equation, and [3,4] for those of diffusion type in Nelson's stochastic mechanics. Consider, for instance, the Landau type equation

$$\partial_t f_t = \frac{1}{2}\mathrm{div}\bigg\{\int_{\mathbb{R}^d} a(\cdot - z)\big(f_t(z)\nabla f_t - f_t\nabla f_t(z)\big)\mathrm{d}z\bigg\}, \tag{1.3}$$

for some reference coefficient $a: \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$. This includes the homogeneous Landau equation where $d = 3$ and

$$a(x) = |x|^{2+\gamma}\bigg(I - \frac{x\otimes x}{|x|^2}\bigg), \quad x \in \mathbb{R}^3 \tag{1.4}$$

for some constant $\gamma \in [-3, 1]$. Landau equation is "grazing collision limit" of the Boltzmann equation. When $\gamma \in [0, 1]$, the existence, uniqueness, regularity estimates, and exponential convergence have been investigated in [5–7] and references within for initial density in $L^1_{s_1} \cap L^2_{s_2}$ for large enough $s_1, s_2 > 0$, where $f \in L^p_s$ means $\int_{\mathbb{R}^3} |f(x)|^p(1 + |x|^2)^{\frac{s}{2}}\mathrm{d}x < \infty$. To describe the solution of (1.3) using stochastic processes, consider the following distribution dependent SDE (DDSDE) for $b = \mathrm{div}a$ and σ such that $\sigma\sigma^* = a$:

$$\mathrm{d}X_t = (b * \mathscr{L}_{X_t})(X_t)\mathrm{d}t + (\sigma * \mathscr{L}_{X_t})(X_t)\mathrm{d}W_t, \tag{1.5}$$

where $\mathscr{L}_\xi$ denotes the distribution of a random variable ξ, and

$$(f * \mu)(x) := \int_{\mathbb{R}^d} f(x - z)\mu(\mathrm{d}z)$$

for a function f and a probability measure μ. By Itô's formula and the integration by parts formula, the distribution density of X_t is a weak solution to (1.3). For the homogeneous Landau equation with $\gamma \in [0, 1]$ and initial distribution density f_0 satisfying

$$\int_{\mathbb{R}^3} f_0(x)\big(f_0(x) + \mathrm{e}^{|x|^\alpha}\big)\mathrm{d}x < \infty \ \text{ for some } \alpha > \gamma, \tag{1.6}$$

the existence and uniqueness of weak solutions to (1.5) has been proved in [9] by an approximation argument using particle systems. This approximation is known as *propagation of chaos* according to Kac [14], see also [11,12] and references within.

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