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A regularity theory for quasi-linear Stochastic PDEs in weighted Sobolev spaces

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Received 5 November 2016; received in revised form 24 May 2017; accepted 7 June 2017

Available online xxxx

Abstract

We study the second-order quasi-linear stochastic partial differential equations (SPDEs) defined on C^1-domains. The coefficients are random functions depending on t, x and the unknown solutions. We prove the uniqueness and existence of solutions in appropriate Sobolev spaces, and in addition, we obtain L_p and Hölder estimates of both the solution and its gradient.

MSC: 60H15; 35R60

Keywords: Nonlinear stochastic partial differential equations; Equations of divergence type; Weighted Sobolev space

1. Introduction

In this article we present a weighted Sobolev space theory of the following stochastic partial differential equation (SPDE):

$$\begin{aligned} du = &\left[D_i\left(a^{ij}(t,x,u)u_{x^j} + b^i(t,x,u)u + f^i\right) + \bar{b}^i(t,x,u)u_{x^i} + c(t,x,u)u + f\right]dt \\ &+ (\nu^k(t,x)u + g^k)dW_t^k, \qquad t \le \tau, \quad x \in \mathcal{O}; \quad u(0,\cdot) = u_0. \end{aligned} \tag{1.1}$$

Here τ is an arbitrary bounded stopping time, $\mathcal{O}$ is a bounded C^1-domain in $\mathbf{R}^d$, and W_t^k $(k = 1, 2, \ldots)$ are independent one-dimensional Wiener processes defined on a probability space

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http://dx.doi.org/10.1016/j.spa.2017.06.006

0304-4149/

Please cite this article in press as: I. Kim, K.-H. Kim, A regularity theory for quasi-linear Stochastic PDEs in weighted Sobolev spaces, Stochastic Processes and their Applications (2017), http://dx.doi.org/10.1016/j.spa.2017.06.006.

$(\Omega, \mathcal{F}, P)$. The indices i and j move from 1 to d, and k runs through $\{1, 2, 3, \ldots\}$. Einstein's summation convention with respect to i, j and k is assumed throughout the article. All the coefficients are random, the coefficients $a^{ij}, b^i, \bar{b}^i, c$ depend also on t, x and the unknown u, and the coefficients ν^k ($k = 1, 2, \ldots$) depend on ω, t, and x. We assume that the coefficients are only measurable with respect to (ω, t), Hölder continuous with respect to x, and Lipschitz continuous with respect to the unknown u.

Let $u(t, x)$ denote the density of diffusing particles at the time t and the location x. Typically, the flux density $\mathbf{F}(t, x)$ is proportional to $-\nabla u$ or more generally to $-\sum_j a^{ij} u_{x^j}$, and the classical heat equation $u_t = D_i(a^{ij} u_{x^j})$ is a consequence of the relation $u_t = -\operatorname{div} \mathbf{F}$. Motivation of studying Eq. (1.1) naturally arises since the diffusion coefficients a^{ij} related to the flux density $\mathbf{F}(t, x)$ can depend also on their point density $u(t, x)$. Our equation is this type general equation with noises and random external forces. The external forces f^i, f, and g are contained in a weighted Sobolev space. More precisely,

$$f^i \in \mathbb{H}^{\gamma_0}_{p,d}(\mathcal{O}, \tau) := L_p(\Omega \times (0, \tau]; H^{\gamma_0}_{p,d}(\mathcal{O})), \;\; f \in \mathbb{H}^{\gamma_0 - 1}_{p,d+p}(\mathcal{O}, \tau), \;\; g \in \mathbb{H}^{\gamma_0}_{p,d}(\mathcal{O}, \tau, l_2),$$

where $p > d + 2$ and $\gamma_0 \in ((d + 2)/p, 1)$. The spaces $H^{\gamma}_{p,d}(\mathcal{O})$ and $\mathbb{H}^{\gamma}_{p,d}(\mathcal{O}, \tau)$ are introduced in Section 2. We only remark that if γ is a natural number then $u \in H^{\gamma}_{p,d}(\mathcal{O})$ iff

$$\rho^{|\alpha|} D^{\alpha} u \in L_p(\mathcal{O}), \quad \forall |\alpha| \le \gamma \qquad (\rho(x) := \operatorname{dist}(x, \partial \mathcal{O})).$$

Under the setting, we prove that Eq. (1.1) has a unique solution in a certain weighted Sobolev space, actually in $\mathfrak{H}^1_{2,d}(\mathcal{O}, \tau) \cap \mathfrak{H}^{1+\gamma_0-\varepsilon}_{p,d,\mathrm{loc}}(\mathcal{O}, \tau)$, $(\varepsilon > 0)$. Using an embedding theorem related to this weighted Sobolev space, we get the following interior Hölder estimates of the solution u and its gradient ∇u: for all constants $\kappa, \kappa_1, \alpha, \beta, \beta_1$ satisfying

$$1/p < \kappa_1 < \kappa < 1/2, \quad 1 - 2\kappa - d/p > 0,$$
$$\alpha < \gamma_0, \quad 1/p < \beta_1 < \beta < 1/2, \quad \alpha - 2\beta - d/p > 0,$$

it holds that for all $t < \tau$ (a.s.),

$$|\rho^{-\varepsilon} u|_{C^{\kappa_1 - 1/p}([0,t], C^{1-2\kappa-d/p-\varepsilon}(\mathcal{O}))} < \infty, \quad \forall \varepsilon \in [0, 1 - 2\kappa - d/p] \tag{1.2}$$

$$|\rho^{\alpha-\varepsilon'} u_x|_{C^{\beta_1 - 1/p}([0,t], C^{\alpha-2\beta-d/p-\varepsilon'}(\mathcal{O}))} < \infty, \quad \forall \varepsilon' \in [0, \alpha - 2\beta - d/p]. \tag{1.3}$$

Note that (1.2) is a much better result than the one without ρ. For instance, (1.2) with $\varepsilon = 1 - 2\kappa - d/p$ implies $\sup_{s \le t} |u(s, x)| = O(\rho^{1-d/p-2\kappa}(x))$. This shows how fast $u(t, x) \to 0$ as $\rho(x) \to 0$. By taking $p \to \infty$ one can make $1 - 2\kappa - d/p$ (the Hölder exponent of u in space variable) as close to 1 as one wishes, and similarly $\kappa_1 - 1/p$ (the Hölder exponent of u in time variable) can be any number in $(0, 1/2)$. Also note that (1.3) implies that u_x is Hölder continuous in (t, x) only interior of $\mathcal{O}$, that is $|u_x|^p_{C^{\beta_1 - 1/p}([0,t], C^{\alpha-2\beta-d/p}(K))} < \infty$ for any compact set $K \subset \mathcal{O}$.

Below we introduce some related results handling divergence or non-divergence type SPDEs whose leading coefficients depend also on the solution u. The 1-dimensional non-divergence type equation

$$du = a(t, x, u)u'' dt + (b(t, x)u' + h(t, x)u) dW_t, \qquad t \le \tau, \quad x \in \mathbf{R}$$

is studied [1] under the assumption that coefficients a, b, h are infinitely differentiable with bounded derivatives. A similar equation

$$du = [a(t, x, u)u'' + f(t, x)]dt + g^k(t, x) dW^k_t \tag{1.4}$$