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Fluctuations of the total number of critical points of random spherical harmonics

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Abstract

We determine the asymptotic law for the fluctuations of the total number of critical points of random Gaussian spherical harmonics in the high degree limit. Our results have implications on the sophistication degree of an appropriate percolation process for modelling nodal domains of eigenfunctions on generic compact surfaces or billiards.

Keywords: Spherical harmonics; Critical points; Kac-Rice formula; Legendre polynomials; Hilb's asymptotics

1. Introduction and main results

1.1. Critical points of random spherical harmonics

It is well-known that the eigenvalues λ of the Laplacian Δ on the 2-dimensional round unit sphere $\mathcal{S}^2$, satisfying the Schrödinger equation

$$\Delta f + \lambda f = 0$$

are of the form $\lambda = \lambda_\ell = \ell(\ell+1)$ for some integer $\ell \geq 1$. For any given eigenvalue λ_ℓ of the above form, the corresponding eigenspace is the $(2\ell+1)$-dimensional space of spherical

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harmonics of degree ℓ; we can choose an arbitrary L^2-orthonormal basis $\{Y_{\ell m}(.)\}_{-\ell\le m\le\ell}$, and consider random eigenfunctions of the form

$$f_\ell(x) = \frac{1}{\sqrt{2\ell+1}}\sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(x), \tag{1.1}$$

where the coefficients $\{a_{\ell m}\}_{-\ell\le m\le\ell}$ are independent, standard Gaussian variables. The random fields

$$\{f_\ell(x),\ x \in \mathcal{S}^2\}$$

are centred Gaussian and the law of f_ℓ in (1.1) is invariant with respect to the choice of $\{Y_{\ell m}\}$. Also, f_ℓ are isotropic, meaning that the probability laws of $f_\ell(\cdot)$ and $f_\ell^g(\cdot) := f_\ell(g\cdot)$ are the same for every rotation $g \in SO(3)$. Here we choose the commonly adopted basis of real valued spherical harmonics

$$Y_{\ell m}(\theta,\varphi) = \begin{cases} \sqrt{2}\,\mathcal{K}_\ell^m \cos(m\varphi) P_\ell^m(\cos\theta), & \text{if } m > 0, \\ \mathcal{K}_\ell^0 P_\ell^0(\cos\theta), & \text{if } m = 0, \\ \sqrt{2}\,\mathcal{K}_\ell^m \sin(-m\varphi) P_\ell^{-m}(\cos\theta), & \text{if } m < 0; \end{cases}$$

where P_ℓ^m are the associated Legendre functions and

$$\mathcal{K}_\ell^m = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-|m|)!}{(\ell+|m|)!}}.$$

By the addition theorem for spherical harmonics [3, Theorem 9.6.3] the covariance function of f_ℓ is given by

$$\mathbb{E}[f_\ell(x) f_\ell(y)] = P_\ell(\cos d(x,y)),$$

where P_ℓ are the usual Legendre polynomials,

$$\cos d(x,y) = \cos\theta_x \cos\theta_y + \sin\theta_x \sin\theta_y \cos(\varphi_x - \varphi_y)$$

is the spherical geodesic distance between x and y, $\theta \in [0,\pi]$, $\varphi \in [0,2\pi)$ are standard spherical coordinates and (θ_x,φ_x), (θ_y,φ_y) are the spherical coordinates of x and y respectively.

Our primary focus is the total number of critical points of f_ℓ

$$\mathcal{N}^c(f_\ell) = \#\{x \in \mathcal{S}^2 : \nabla f_\ell(x) = 0\}.$$

It is known [19,11] that, as $\ell \to \infty$, the expected total number of critical points $\mathcal{N}^c(f_\ell)$ is asymptotic to

$$\mathbb{E}[\mathcal{N}^c(f_\ell)] = \frac{2}{\sqrt{3}}\ell^2 + O(1).$$

An upper bound for the variance of the number of critical points $\mathcal{N}^c(f_\ell)$ was also derived [11]:

$$\mathrm{Var}(\mathcal{N}^c(f_\ell)) = O\left(\ell^{\frac{5}{2}}\right);$$

in fact, it is likely that the same method yields the stronger result

$$\mathrm{Var}(\mathcal{N}^c(f_\ell)) = O(\ell^2 \log \ell).$$

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