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Dynamic uniqueness for stochastic chains with unbounded memory

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Abstract

We say that a probability kernel exhibits dynamic uniqueness (DU) if all the stochastic chains starting from a fixed past coincide on the future tail σ-algebra. Our first theorem is a set of properties that are pairwise equivalent to DU which allow us to understand how it compares to other more classical concepts. In particular, we prove that DU is equivalent to a weak-ℓ^2 summability condition on the kernel. As a corollary to this theorem, we prove that the Bramson–Kalikow and the long-range Ising models both exhibit DU if and only if their kernels are ℓ^2 summable. Finally, if we weaken the condition for DU, asking for coincidence on the future σ-algebra for almost every pair of pasts, we obtain a condition that is equivalent to β-mixing (weak-Bernoullicity) of the compatible stationary chain. As a consequence, we show that a modification of the weak-ℓ^2 summability condition on the kernel is equivalent to the β-mixing of the compatible stationary chain.

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1. Introduction

Stochastic chains with unbounded memory (SCUM's in what follows) are generalizations of Markov chains in which the dependence on the past can be unbounded. Similar to a Markov chain, a SCUM consists of an initial condition (a probability measure on the infinite past) and a probability kernel (set of transition probabilities) that defines the probability of appearance of a symbol at each step given its past. Given that these chains can have dependences on an unbounded part of the past, a natural question to ask is how the future depends on the choice of the past symbols. In physics, the interest in such questions lies in the possibility that those dependences could model "phase transition" phenomena. In mathematics, it is a question of whether different initial conditions result on processes that assign different probabilities to the events of interest.

Traditionally, uniqueness or phase transition of stochastic chains has been studied by considering the uniqueness/non-uniqueness of equilibrium states, often called g-measures [6,12,16,18,25,27,33]. In this framework, we ask if the probability of the cylinder events depends on the choice of the initial conditions far away in the past. The kernel exhibits phase transition if this probability depends on the initial condition, otherwise it satisfies uniqueness. The analogy with Gibbs measure is clear: the initial condition (past) plays the role of the boundary condition, the transition probabilities are the specifications, and g-measures are analogous to Gibbs measures [12].

Despite the success of this approach, the theory of g-measures naturally excludes some important questions concerning SCUM's. For instance, only their past is fixed for stochastic chains, in contrast to the boundary conditions in Gibbsian theory, where both the past and future are fixed. Furthermore, when considering the events of interest for detecting the presence of a phase transition, it is not obvious why we should consider only the cylinder events. The future asymptotic events, that do not depend on the values of a chain on a finite set of coordinates, are also natural candidates for detecting phase transition for stochastic chains. Hence, we ask the following questions: Is the probability of the asymptotic events of SCUM's the same for any choice of the initial condition? If not, what conditions on the kernel do imply that the probabilities of the asymptotic events of the chains starting from different initial conditions are equal/different? Such questions have not been considered in the literature of g-measures.

This article proposes a starting point to study these questions, introducing the notion of dynamic uniqueness (DU) and dynamic phase transition (DPT), suited to study the properties of non-stationary chains starting with different pasts. The basic idea is the following. Given a kernel and the corresponding SCUM's starting with distinct fixed pasts, we want to study their measures on the future tail σ-algebra. The events on this tail σ-algebra (tail events) can have the physical interpretation of asymptotic events. We say that a kernel exhibits a DPT if there exist two different pasts for which the corresponding measures on the tail events disagree. Conversely, we say that there is DU if the measures on the tail σ-algebra agree for all pasts.

A detailed discussion of the results will be given as we state them in Section 3, but let us just give a summary here. We first prove several equivalent criteria for DU (Theorem 1). In particular, we show that DU is equivalent to a weak-ℓ^2 criterion and to convergence in total variation. We also prove that DU is strictly stronger than equilibrium uniqueness and, in fact, strictly stronger than weak-Bernoullicity (also called absolute regularity or β-mixing). Finally, we prove that weak-Bernoullicity is equivalent to the requirement that our weak-ℓ^2 criterion holds almost surely.

The paper is organized as follows. Section 2 introduces the notations and the basic definitions concerning SCUM's. Section 3 contains the main results, as well as their discussions. For the