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Mean-field limit of generalized Hawkes processes

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Received 20 October 2015; received in revised form 25 January 2017; accepted 4 February 2017

Available online xxxx

Abstract

We generalize multivariate Hawkes processes mainly by including a dependence with respect to the age of the process, i.e. the delay since the last point.

Within this class, we investigate the limit behaviour, when n goes to infinity, of a system of n mean-field interacting age-dependent Hawkes processes. We prove that such a system can be approximated by independent and identically distributed age dependent point processes interacting with their own mean intensity. This result generalizes the study performed by Delattre et al. (2016).

In continuity with Chevallier et al. (2015), the second goal of this paper is to give a proper link between these generalized Hawkes processes as microscopic models of individual neurons and the age-structured system of partial differential equations introduced by Pakdaman et al. (2010) as macroscopic model of neurons.

MSC: 60G55; 60F05; 60G57; 92B20

Keywords: Hawkes process; Mean-field approximation; Interacting particle systems; Renewal equation; Neural network

1. Introduction

In the recent years, the self-exciting point process known as the Hawkes process [29] has been used in very diverse areas. First introduced to model earthquake replicas [32] or [41] (ETAS

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http://dx.doi.org/10.1016/j.spa.2017.02.012
0304-4149/

model), it has been used in criminology to model burglary [39], in genomic data analysis to model occurrences of genes [27,50], in social networks analysis to model viewing or popularity [3,14], as well as in finance [1,2]. We refer to [35] or [56] for more extensive reviews on applications of Hawkes processes. A univariate (nonlinear) Hawkes process is a point process N admitting a stochastic intensity of the form

$$\lambda_t = \Phi\left(\int_0^{t-} h(t-z)N(dz)\right), \tag{1}$$

where $\Phi : \mathbb{R} \to \mathbb{R}_+$ is called the intensity function, $h : \mathbb{R}_+ \to \mathbb{R}$ is called the self-interaction function (also called exciting function or kernel function in the literature) and $N(dz)$ denotes the point measure associated with N. We refer to [53–55,57,58] for recent papers dealing with nonlinear Hawkes process.

Such a form of the intensity is motivated by practical cases where all the previous points of the process may impact the rate of appearance of a new point. The influence of the past points is formulated in terms of the delay between those past occurrences and the present time, through the weight function h. In the natural framework where h is non-negative and Φ increasing, this choice of interaction models an excitatory phenomenon: each time the process has a jump, it excites itself in the sense that it increases its intensity and thus the probability of finding a new point. A classical case is the *linear* Hawkes process for which h is non-negative and $\Phi(x) = \mu + x$ where μ is a positive constant called the spontaneous rate. Note however that Hawkes processes can also describe inhibitory phenomena. For example, the function h may take negative values, Φ being the positive part modulo the spontaneous rate μ, i.e. $\Phi(x) = \max(0, \mu + x)$.

Multivariate Hawkes processes consist of multivariate point processes $(N^1, \ldots, N^n)$ whose intensities are respectively given for $i = 1, \ldots, n$ by

$$\lambda_t^i = \Phi_i\left(\sum_{j=1}^{n}\int_0^{t-} h_{j\to i}(t-z)N^j(dz)\right), \tag{2}$$

where $\Phi_i : \mathbb{R} \to \mathbb{R}_+$ is the intensity function associated with the particle i and $h_{j\to i}$ is the *interaction function* describing the influence of each point of N^j on the appearance of a new point onto N^i, via its intensity λ^i.

When the number of interacting particles is huge (as, for instance, financial or social networks agents), one may be willing to let the number of particles goes to infinity. This is especially true for multivariate Hawkes processes subject to mean-field interactions. In such a case, we may indeed expect propagation of chaos, namely the particles are expected to become asymptotically independent, provided that they start from independent and identically distributed (i.i.d.) initial conditions and submitted to i.i.d. sources of noise. Mean-field type interactions involve some homogeneity and some symmetry through coefficients that depend upon the empirical measure of the processes: In the limit regime, the coefficients depend upon the common asymptotic distribution of the particles, which satisfies nonlinear dynamics, sometimes called of McKean–Vlasov type.

The study of mean-field situations for Hawkes processes was initiated by Delattre et al. [17] by considering the following form of intensity

$$\lambda_t^i = \Phi\left(\frac{1}{n}\sum_{j=1}^{n}\int_0^{t-} h(t-z)N^j(dz)\right), \tag{3}$$