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Stability problems for Cantor stochastic differential equations

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Abstract

We consider driftless stochastic differential equations and the diffusions starting from the positive half line. It is shown that the Feller test for explosions gives a necessary and sufficient condition to hold pathwise uniqueness for diffusion coefficients that are positive and monotonically increasing or decreasing on the positive half line and the value at the origin is zero. Then, stability problems are studied from the aspect of Hölder-continuity and a generalized Nakao–Le Gall condition. Comparing the convergence rate of Hölder-continuous case, the sharpness and stability of the Nakao–Le Gall condition on Cantor stochastic differential equations are confirmed. Furthermore, using the Malliavin calculus, we construct a smooth solution to degenerate second order Fokker–Planck equations under weak conditions on the coefficients.

MSC: primary 60J55; secondary 41A25

Keywords: Stochastic differential equation; Stability problems; Convergence rate; Local time

1. Introduction

Given an interval $I = (0, \infty)$ and a real-valued Borel function $\sigma : \mathbb{R} \to \mathbb{R}$, let us consider

$$X(t) - X(0) = \int_0^t \sigma(X(s))dB_s, \ X(0) = x_0 \in I. \tag{1}$$

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Skorokhod [22] showed that if σ is continuous that the stochastic differential equations (SDEs) have a weak solution on a filtered probability space. Moreover, if σ^{-2} is integrable over a neighborhood of x for all $x \in I$, it follows from Engelbert and Schmidt [8] that the weak solution exists law up to stopping time S, satisfying $\mathbb{P}(S = \inf\{t \geq 0 : X(t) = 0\}) = 1$, where the notation $\inf \emptyset$ means $+\infty$.

Therefore, we are interested in studying the behavior of the solution around the boundary point 0, which has an identity as scale function and speed measure. In fact, we present the following proposition.

Proposition 1.1. *Let c be in I and σ be a real-valued Borel function that monotonically increases or decreases in I such that $(\sigma(x))^2 > 0$ for every $x \in I$. Then, pathwise uniqueness holds for (1) if*

$$\mu(0+) = \lim_{x \downarrow 0} \int_x^c \frac{c-y}{(\sigma(y))^2} dy = \infty.$$

Moreover, suppose that $\sigma(0) = 0$. Then, pathwise uniqueness holds for (1) if and only if $\mu(0+) = \infty$.

We first investigate the so-called stability problem proposed by Stroock and Varadhan [23]. More precisely, we consider convergence that for Borel measurable functions $\{\sigma_n\}_{n \in \mathbb{N}}$, the series of solutions $\{X_n\}$,

$$X_n(t) - X_n(0) = \int_0^t \sigma_n(X_n(s)) dB_s, \ \ X_n(0) = x_0 \in I,$$

converges to X when σ_n tend to σ in the following sense,

$$\Delta_n := \sup_{x \in \mathbb{R}} |\sigma(x) - \sigma_n(x)|.$$

By Proposition 1.1, for the Hölder-continuous diffusion coefficients with exponent $\mathrm{H} \in [0, 1]$, pathwise uniqueness holds if and only if $\mathrm{H} \in [\frac{1}{2}, 1]$. Applying the so-called Yamada–Watanabe method [24] (cf. and [5]) to the stability problems in Section 3, we prove the following theorem:

Theorem 1.1. *Let $\mathrm{H} \in [0, 1]$. Suppose that $\|\sigma\|_\infty = \sup_{x \in \mathbb{R}} |\sigma(x)| < \infty$ and there exists some $c_{\mathrm{H}} > 0$ such that for all $x, y \in \mathbb{R}$ satisfying $|x - y| \leq 1$, $|\sigma(x) - \sigma(y)| \leq c_{\mathrm{H}} |x - y|^{\mathrm{H}}$. Moreover, suppose that X_n is a strong solution for all $n \in \mathbb{N}$. Then, for any $T > 0$, there exists a $C_2(T, \mathrm{H}) > 0$ dependent on H and T such that for every $n \in \mathbb{N}$ satisfying $\Delta_n < 2^{-\mathrm{H}}$,*

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} |X(t) - X_n(t)|\right]^2 \leq C_2(T, \mathrm{H}) \times \begin{cases} (-\log \Delta_n)^{-\frac{1}{2}} & \text{if } \mathrm{H} = \dfrac{1}{2}, \\ \Delta_n^{1-(1/2\mathrm{H})} & \text{if } \mathrm{H} \in \left(\dfrac{1}{2}, 1\right]. \end{cases}$$

For Euler–Maruyama's schemes for SDEs with Lipschitz continuous coefficients, it is known that Euler–Maruyama's convergence rate is $n^{-\frac{1}{2}}$, where the uniform partition is given by $\{iT/n : i = 0, 1, \ldots, n\}$. The distribution of the limit law of the error process for Euler–Maruyama's approximation process was established through many contributions. We refer the reader to [18, Section 5], [1]; for more details, see the excellent survey [15].

Because the result obtained from this weak convergence does not provide full information regarding the rate of convergence such that $|X(t) - X_n(t)|$, other efforts have recently been

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