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On purely discontinuous additive functionals of subordinate Brownian motions

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Abstract

Let $A_t = \sum_{s \le t} F(X_{s-}, X_s)$ be a purely discontinuous additive functional of a subordinate Brownian motion $X = (X_t, \mathbb{P}_x)$. We give a sufficient condition on the non-negative function F that guarantees that finiteness of A_∞ implies finiteness of its expectation. This result is then applied to study the relative entropy of $\mathbb{P}_x$ and the probability measure induced by a purely discontinuous Girsanov transform of the process X. We prove these results under the weak global scaling condition on the Laplace exponent of the underlying subordinator.

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1. Introduction

Let $X = (X_t, \mathbb{P}_x)$ be an isotropic Lévy process in $\mathbb{R}^d$, $d \ge 1$. For a non-negative, bounded and symmetric function $F : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty)$, define the purely discontinuous additive functional $A_t := \sum_{0<s\le t} F(X_{s-}, X_s)$. It is often important to understand when does the finiteness of the additive functional at infinity, $A_\infty < \infty$, imply the finiteness of the expectation of A_∞. To be more precise, we will be interested in sufficient conditions on the function F such that $\mathbb{P}_x(A_\infty < \infty)$ for all $x \in \mathbb{R}^d$ implies that $\mathbb{E}_x A_\infty < \infty$ for all $x \in \mathbb{R}^d$. In the case of an isotropic

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α-stable process X this question was investigated in [14], cf. Theorem 4.15. The result of that theorem was further used to study the relative entropy of $\mathbb{P}_x$ and the probability measure induced by a purely discontinuous Girsanov transform of the stable process X, see [14, Theorem 1.2].

The goal of this paper is to extend the results of [14] from the stable process to a rather large class of subordinate Brownian motions. Instead of the strict scaling property enjoyed by the stable process, we will impose as a substitute the weak global scaling condition. More precisely, let $W = (W_t, \mathbb{P}_x)$ be a d-dimensional Brownian motion running twice as fast as the standard Brownian motion, $d \geq 1$, and let $S = (S_t)_{t\geq 0}$ be an independent subordinator with the Laplace exponent ϕ. The process $X = (X_t, \mathbb{P}_x)$ defined by $X_t := W_{S_t}$ is called a subordinate Brownian motion. It is an isotropic Lévy process with the characteristic exponent $\psi(\xi) = \phi(|\xi|^2)$. Since any Lévy process is completely characterized by its characteristic exponent, we will, without loss of generality, throughout the paper assume that the subordinate Brownian motion $X = (\Omega, \mathcal{M}, \mathcal{M}_t, \theta_t, X_t, \mathbb{P}_x)$ is defined on the Skorokhod path space $\Omega = D([0,\infty), \mathbb{R}^d)$ of cadlag functions $\omega : [0,\infty) \to \mathbb{R}^d$ with $X_t(\omega) = \omega(t)$ being projections, $\mathcal{M} = \sigma\left(\cup_{t\geq 0}\mathcal{M}_t\right)$, and the shift $(\theta_t\omega)(s) = \omega(s+t)$.

Our main assumption on the Laplace exponent ϕ is the following global scaling condition: There exist constants $0 < \delta_1 \leq \delta_2 < 1 \wedge \frac{d}{2}$ and $a_1, a_2 > 0$ such that

$$a_1\lambda^{\delta_1} \leq \frac{\phi(\lambda x)}{\phi(x)} \leq a_2\lambda^{\delta_2}, \quad \lambda \geq 1, x > 0. \tag{1.1}$$

Recall that the Laplace exponent ϕ is a Bernstein function satisfying $\phi(0) = 0$, which implies the representation

$$\phi(\lambda) = b\lambda + \int_{(0,\infty)} (1 - e^{-\lambda t})\,\mu(dt), \quad \lambda > 0,$$

where μ is the Lévy measure of ϕ. Under (1.1) we have $b = 0$ and $\phi(s) \geq a_2^{-1}s^{\delta_2}$ for $s \in (0,1)$. The latter implies that X is transient, cf. (3.1). The function ϕ is called a complete Bernstein function if $\mu(dt) = \mu(t)dt$ with a completely monotone density $\mu(t)$, cf. [13]. For simplicity, let $\Phi(s) := \phi(s^{-2})^{-1}$.

Let $F : \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty)$ be symmetric and bounded. The next condition on F will be crucial for our results: Assume that there are constants $C > 0$ and $\beta > 1$ such that

$$F(x,y) \leq C\frac{\Phi(|x-y|)^\beta}{1 + \Phi(|x|)^\beta + \Phi(|y|)^\beta}, \qquad \text{for all } x, y \in \mathbb{R}^d. \tag{1.2}$$

In the context of elliptic diffusion, the analog of (1.2) is sometimes called the Fuchsian condition, cf. [1]. The main result of this paper is the following theorem.

Theorem 1.1. *Suppose that X is the subordinate Brownian motion via the subordinator whose Laplace exponent is a complete Bernstein function and satisfies (1.1). Let $\beta > 1$ and assume that $F : \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty)$ is bounded, symmetric and satisfies condition (1.2). Let $A_t^F = \sum_{0<s\leq t} F(X_{s-}, X_s)$. If $\mathbb{P}_x(A_\infty^F < \infty) = 1$ for all $x \in \mathbb{R}^d$, then $\sup_{x\in\mathbb{R}^d}\mathbb{E}_x[A_\infty^F] < \infty$.*

It is easy to see, cf. [14, Remark 4.16] in the stable case, that there exists F satisfying (1.2) such that $\mathbb{E}_x[A_\infty] = \infty$. Of course, in this case it cannot hold that $\mathbb{P}_x(A_\infty < \infty) = 1$. On the other hand, condition (1.2) is almost necessary for the validity of Theorem 1.1. We first need the following definition.

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